Overview on pyramid wavefront sensor: forward models, reconstruction algorithms, practical issues

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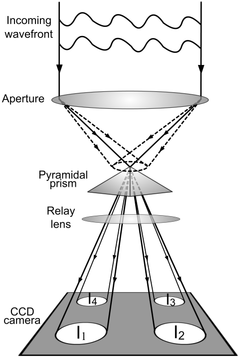
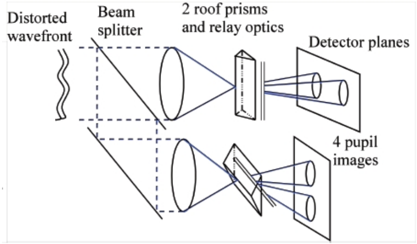
WaveFront Sensing in the VLT/ELT era II,
Padova, October 2-4, 2017

Outline

Outline

Pyramid and Roof WFS





Credit: C. Vérinaud

$$S_x(x,y) = \frac{[I_1(x,y) + I_2(x,y)] - [I_3(x,y) + I_4(x,y)]}{I_0}$$

$$S_y(x,y) = \frac{[I_1(x,y) + I_4(x,y)] - [I_2(x,y) + I_3(x,y)]}{I_0}$$

I_0 – average intensity per subaperture.

Inverse problem & Remarks

- Task: to reconstruct the unknown wavefront ϕ from non- / modulated pyramid WFS data S_x, S_y

$$S_x = P_x\phi, \qquad S_y = P_y\phi$$

- Forward operators P_x, P_y are nonlinear singular integral operators
- Details omitted (e.g., S_x only)
- Any modulation meant; α_λ will denote the modulation parameter

$$\alpha_\lambda = \frac{2\pi\alpha}{\lambda} = \frac{2\pi r}{D}, \qquad d \in \mathbb{R}_+$$

- Omit aperture sometimes (for clarity)
- Finite sampling
- From simple approximate to complicated models

Outline

1. Introduction
2. **Linear models and reconstructors**
   - Roof WFS: linearized and simplified model
   - Algorithms in closed-loop simulations: quality, speed and spiders
   - Extension of algorithms to other linear models
3. Non-linear models and reconstructors
   - Roof WFS: nonlinear transmission mask model
   - Algorithms in closed-loop simulations: quality and speed
   - Extenstion of algorithms to other non-linear models
4. Summary

Roof WFS: linearized and simplified operator $\boldsymbol{R}_s^{\{n,l,c\}}$

$$S_x^{\{n,l,c\}} = \boldsymbol{R}_s^{\{n,l,c\}} \phi$$

$$(\boldsymbol{R}_s^{\{n,l,c\}} \phi)(x,y) = \frac{1}{\pi} \int_{-X(y)}^{X(y)} \frac{\phi(x',y) k_{\{n,l,c\}}(x'-x)}{x-x'} dx' = \left[\phi * m_x^{\{n,l,c\}}\right](x,y)$$

$$m_x^{\{n,l,c\}}(x,y) := \frac{k_{\{n,l,c\}}(x)\delta(y)}{\pi x}$$

$$k_n(x) = 1$$

$$k_l(x) = \mathrm{sinc}(\alpha_\lambda(x))$$

$$k_c(x) = J_0(\alpha_\lambda(x))$$

J_0 – the zero-order Bessel function of the first kind.

Inversion of $R_s^{\{n,l,c\}}$ in spatial domain

- Inversion of Finite Hilbert transform: (modulation 0 only, i.e., R_s^n)
  - Finite Hilbert Transform Reconstructor (FHTR) [1]
  - Singular Value Type Reconstructor (SVTR) [2] $\rightarrow$ SVTR for $R_s^{\{l,c\}}$?
- Iterative algorithms: adjoint operator
  - Conjugate Gradient for the Normal Equation (CGNE) [1,3,4]
  - Steepest Descent (SD) [3,4]
  - Pyramid Kaczmarz Iteration (PKI) [3,4]

[1] I. Shatokhina, “Fast wavefront reconstruction algorithms for extreme adaptive optics,” Ph.D. thesis (Johannes Kepler University Linz, 2014).

[2] V. Hutterer, R. Ramlau, Wavefront Reconstruction from Non-modulated Pyramid Wavefront Sensor Data using a Singular Value Type Expansion, Inverse Problems, submitted.

[3] V. Hutterer, R. Ramlau, Iu. Shatokhina, Real-time AO with pyramid wavefront sensors: Theoretical analysis of pyramid forward model, in preparation.

[4] V. Hutterer, R. Ramlau, Iu. Shatokhina, Real-time AO with pyramid wavefront sensors: Accurate wavefront reconstruction with iterative methods, in preparation.

FHTR & SVTR

Invert

$$R_s^n = T_x\phi = [\phi * m_x^n]$$

$$s_x^n = R_s^n$$

$$\phi = T_x^{-1} s_x$$

FHTR:

$$\phi(x) = -\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x^2}{1-x'^2}}\frac{s_x(x')}{x-x'}dx'$$

SVTR:

singular value type system $(\sigma_k, f_k, g_k)_{k\geq 0}$,
f_k, g_k – weighted Chebyshev polynomials,
$\sigma_k = 1 \quad \forall k$

$$\phi(x,y) = -2\sum_{k=0}^{\infty}\frac{1}{\sigma_k}\langle s_x(\cdot, y), g_k\rangle_\omega f_k(x).$$

Iterative algorithms: CGNE & SD & PKI

- Well-known (in applied mathematics) iterative methods
- Application of adjoint operators

$$\left(\left(\boldsymbol{R}_s^{\{n,c,l\}}\right)^* \Psi\right)(x,y) = -\frac{1}{\pi}\, p.v. \int\limits_{\Omega_y} \frac{\Psi(x',y)\cdot k^{\{n,c,l\}}(x'-x)}{x'-x}\, dx',$$

- Due to discretization largely precomputed $\longrightarrow$ fast!

Algorithm: Landweber-Kaczmarz iteration

choose Φ_0, set attenuation coefficients β_1, β_2
for $i = 1, \ldots K$ do
    $\Phi_{i,0} = \Phi_{i-1}$
    $\Phi_{i,1} = \Phi_{i,0} + \beta_1 \boldsymbol{R}_x^* \left(s_x - \boldsymbol{R}_x \phi_{i,0}\right)$
    $\Phi_{i,2} = \Phi_{i,1} + \beta_2 \boldsymbol{R}_y^* \left(s_y - \boldsymbol{R}_y \phi_{i,1}\right)$
    $\Phi_i = \Phi_{i,2}$
endfor

Inversion of $\boldsymbol{R}_s^{\{n,l,c\}}$ in Fourier domain

Fourier domain representation of $\boldsymbol{R}_s^{\{n,l,c\}}\phi$:

$$(\mathcal{F}_x S_x^{\{n,l,c\}})(u) = \left[(\mathcal{F}_x \phi)(u) \cdot g_{\{n,l,c\}}(u)\right]$$

$$g_{\{n,l,c\}}(u) = \left(\mathcal{F}_x m_x^{\{n,l,c\}}\right)(u)$$

- Fourier domain based algorithms:
  - Preprocessed Cumulative Reconstructor with domain Decomposition (P-CuReD) [1,2]
  - Convolution with the Linearized Inverse Filter (CLIF) [2,3]
  - Pyramid Fourier Transform Reconstructor (PFTR) [2,3]

[1] Iu. Shatokhina, A. Obereder, R. Rosensteiner, R. Ramlau. Preprocessed cumulative reconstructor with domain decomposition: a fast wavefront reconstruction method for pyramid wavefront sensor, Applied Optics 52(12), 2640-2652 (2013).

[2] I. Shatokhina, R. Ramlau. Convolution and Fourier transform based reconstructors for pyramid wavefront sensor, Applied Optics 56(22), 6381-6390 (2017).

[3] I. Shatokhina, “Fast wavefront reconstruction algorithms for extreme adaptive optics,” Ph.D. thesis (Johannes Kepler University Linz, 2014).

P-CuReD & CLIF & PFTR

$$(\mathcal{F}_x S_x)(u) = (\mathcal{F}_x \phi)(u) \cdot g(u)$$

P-CuReD:

$$(\mathcal{F}_x S)^{SH}_{pyr}(u) = (\mathcal{F}_x \phi)(u) \cdot g^{SH}_{pyr}(u)$$

$$(\mathcal{F}_x S_{SH})(u) = (\mathcal{F}_x S_{pyr})(u) \cdot g_{SH/pyr}(u)$$

$$g_{SH/pyr}(u) := \frac{g_{SH}(u)}{g_{pyr}(u)}$$

$$S_{SH}(x) = (S_{pyr} * \underbrace{\mathcal{F}_x^{-1} g_{SH/pyr}}_{p_{SH/pyr}})(x)$$

$$\phi(x, y) = CuReD(S_{SH})$$

PFTR:

$$(\mathcal{F}_x \phi)(u) = (\mathcal{F}_x S_x)(u) \cdot g^{-1}(u)$$

$$\phi(x, y) = \left(\mathcal{F}_x^{-1} \left[(\mathcal{F}_x S_x)(u) \cdot g^{-1}(u)\right]\right)(x, y)$$

CLIF:

$$\phi(x, y) = \left[S_x * \left(\mathcal{F}_x^{-1} g^{-1}\right)\right](x, y)$$

$$p(x, y) = \left(\mathcal{F}_x^{-1} g^{-1}\right)(x)\, \delta(y)$$

$$\phi(x, y) = [S_x * p]\,(x, y)$$

Considered AO systems

XAO (EPICS on ELT)

- aim: direct imaging of exoplanets
- $D = 42$ m telescope
- pyramid WFS with 200x200 subapertures,
  with circular modulation
- DM update 3000 times per second!
- time for reconstruction: 0.3 ms

SCAO (METIS on ELT)

- $D = 37$ m telescope
- pyramid WFS with 74x74 subapertures,
  with circular modulation
- DM update 500-1000 times per second!
- time for reconstruction: 1-2 ms

Comparison of quality: linear algorithms

Algorithm	Quality in end-to-end simulations (OCTOPUS)			
	METIS mod 0	METIS mod 4	XAO mod 0	XAO mod 4
Photon flux	10000 ph/pix/it	10000 ph/pix/it	50 ph/pix/it	50 ph/pix/it
Frame rate	1kHz	1kHz	3kHz	3kHz
Matrix-Vector Multiplication (MVM)	$\approx$ 0.62 [1]	0.80 [2]		0.96
MMSE (YAO)		0.89 [3]		
Preprocessed CuReD (P-CuReD)		0.89	0.91	0.96
Conv. with Linearized Inverse Filter (CLIF)			0.88	0.94
Pyramid FTR (PFTR)			0.88	0.94
Finite Hilbert Transform Rec. (FHTR)			0.85	
Singular Value Type Reconstructor (SVTR)	0.77		0.88	
Steepest Descent (SD)	$\geq$ 0.79		0.90	
Pyramid Kaczmarz Iteration (PKI)	0.81		0.92	

[1] M. Le Louarn et. al., Latest AO simulation results for the E-ELT, poster AO4ELT5.

[2] Results provided by ESO.

[3] MMSE reconstructor in YAO, results provided by Stefan Hippler.

Comparison of complexities: linear methods

Algorithm	Modulation no	Modulation small	Modulation large	Complexity	Remarks
Matrix-Vector Multiplication (MVM)	+	+	+	$O(n^2)$	baseline;
Fourier Transform Reconstructor (FTR)	–	–	+	$O(n\log n)$	geometrical model
Preprocessed CuReD (P-CuReD)	+	+	+	$O(n)$	Fourier domain based
Conv. with Linearized Inverse Filter (CLIF)	+	+	+	$O(n^{3/2})$	(iteartive)
Pyramid FTR (PFTR)	+	+	+	$O(n\log n)$	
Finite Hilbert Transform Rec. (FHTR)	+	–	–	$O(n^{3/2})$	inversion of finite
Singular Value Type Reconstructor (SVTR)	+	–	–	$O(n^{3/2})$	Hilbert transform
Conjugate Gradient for Normal Eq. (CGNE)	+	+	+	$O(n^{3/2})$	iterative algorithms,
Steepest Descent (SD)	+	+	+	$O(n^{3/2})$	adjoint operators
Pyramid Kaczmarz Iteration (PKI)	+	+	+	$O(n^{3/2})$	

Comparison of computational load: linear methods

Algorithm		Number of operations in XAO setting	
MVM	$4n_a n$	3.4120e+09	100%
P-CuReD	$(4c-2)n + 20n$	1.3248e+06	0.0388%
CLIF	$4n\sqrt{n} + n$	1.9579e+07	0.5738%
SD($K = 4$)	$K \cdot (12n\sqrt{n} + 12n + 4)$	$4 \cdot 5.8996e + 07$	$4 \cdot 1.73\% = 6.92\%$
PKI($K = 5$)	$K \cdot (8n\sqrt{n} + 2n)$	$5 \cdot 3.9158e + 07$	$5 \cdot 1.15\% = 5.75\%$

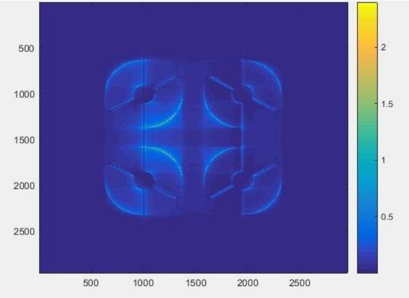
$n_a = 29618$, $n = 28800$, $c = 7$

Reconstruction in presence of spiders

- Residual segmented piston ⟶ low LE
- There are possibilities to control differential piston
  - four methods
  - some provide acceptable quality
  - extremely fast! Add 6x2n_a FLOPs
- Poke matrix inversion ⟶ see talk by A. Obereder tomorrow @ 12.40
  ”Keep it simple – Poke Matrix Inversion for a (stable) piston segment reconstruction”
- Split approaches ⟶ see talk by V. Hutterer tomorrow @ 12.00
  ”Direct piston reconstruction approaches to control segmented ELT-mirrors”

Important questions

- Spiders – theoretical understanding / explanation
  - sign (not possible from linearized roof sensor models)
  - full pyramid model, take interference terms into account?
  - reconstruction of pistons from intensities $I_{\{1,2,3,4\}}$?
    - identify segments between which piston jumps occur
    - criteria to identify if the sign of piston between neighbouring segments is the same, or the opposite
    - criteria for piston sign – work in progress
- What is the best possible reconstruction quality?



Other linear models

Model	Roof-WFS	Pyramid WFS
Linear simplified	$\boldsymbol{R}_s^{\{n,l,c\}}$	$\boldsymbol{P}_s^{\{n,l,c\}}$
Linear	$\boldsymbol{R}_l^{\{n,l,c\}}$	$\boldsymbol{P}_l^{\{n,l,c\}}$

$$(\boldsymbol{R}_s^{\{n,l,c\}}\phi)(x,y) = \frac{1}{\pi}\int_{-X(y)}^{X(y)} \frac{\phi(x',y)k_{\{n,l,c\}}(x'-x)}{x-x'}dx'$$

$$(\boldsymbol{R}_l^{\{n,l,c\}}\phi)(x,y) = (\boldsymbol{R}_s^{\{n,l,c\}}\phi)(x,y) - \phi(x,y)(\boldsymbol{R}_s^{\{n,l,c\}}1)(x,y)$$

$$(\boldsymbol{P}_s^{\{n,l,c\}}\phi)(x,y) = (\boldsymbol{R}_s^{\{n,l,c\}}\phi)(x,y) - \frac{1}{\pi^3}\int_{-X(y)}^{X(y)}\int_{-Y(x)}^{Y(x)}\int_{-Y(x)}^{Y(x)} \frac{\phi(x',y')p_{\{n,c\}}(x'-x,y'-y'')}{(x-x')(y-y')(y-y'')}dy''dy'dx'$$

$$p_c(\tilde{x},\tilde{y}) := \frac{1}{T}\int_{-T/2}^{T/2} \cos[\alpha_\lambda \tilde{x}\sin(2\pi t/T)]\cos[\alpha_\lambda \tilde{y}\cos(2\pi t/T)]dt$$

$$(\boldsymbol{P}_l^{\{n,l,c\}}\phi) = (\boldsymbol{R}_l^{\{n,l,c\}}\phi) - \frac{1}{\pi^3}\int_{-X(y)}^{X(y)}\int_{-Y(x)}^{Y(x)}\int_{-Y(x)}^{Y(x)} \frac{[\phi(x',y')-\phi(x,y'')]p_{\{n,c\}}(x'-x,y'-y'')}{(x-x')(y-y')(y-y'')}dy''dy'dx'$$

i-FHTR, i-SVTR; i-PFTR, i-CLIF; CGNE, SD, PKI

Other linear models – Fourier domain representation

Fourier domain representation of

- $\boldsymbol{R}_s^{\{n,l,c\}}\phi$: P-CuReD, PFTR, CLIF

$$(\mathcal{F}_x S_x^{\{n,l,c\}})(u) = \left[(\mathcal{F}_x\phi)(u)\cdot g_{\{n,l,c\}}(u)\right]$$

$$g_{\{n,l,c\}}(u) = \left(\mathcal{F}_x m_x^{\{n,l,c\}}\right)(u)$$

- $\boldsymbol{R}_l^{\{n,l,c\}}\phi$: i-PFTR, i-CLIF

$$(\mathcal{F}_x S_x^{\{n,l,c\}})(u) = \left[(\mathcal{F}_x\phi)(u)\cdot g_{\{n,l,c\}}(u)\right] - (\mathcal{F}_x\phi)(u) * \left[(\mathcal{F}_x\mathcal{X}_{\Omega_y\times\Omega_x})(u)\cdot g_{\{n,l,c\}}(u)\right]$$

- $\boldsymbol{P}_s^{\{n,l,c\}}$
  - Non-modulated case: i-PFTR, i-CLIF

$$\begin{aligned}(\mathcal{F}_{xy} S_x^n) &= (\mathcal{F}_{xy}\phi)\cdot(\mathcal{F}_{xy} m_x^n) \\ &- \left[(\mathcal{F}_{xy}\phi)\cdot(\mathcal{F}_{xy} m_{xy}^n)\right] * \left[(\mathcal{F}_{xy}\mathcal{X}_{\Omega_y\times\Omega_x})\cdot(\mathcal{F}_{xy} m_y^n)\right]\end{aligned}$$

  - Modulated case: ?
- $\boldsymbol{P}_l^{\{n,l,c\}}$

Extension of algorithms to other linear models

Model	Roof-WFS	Pyramid WFS
Linear simplified	$\boldsymbol{R}_s^{\{n,l,c\}}$	$\boldsymbol{P}_s^{\{n,l,c\}}$
Linear	$\boldsymbol{R}_l^{\{n,l,c\}}$	$\boldsymbol{P}_l^{\{n,l,c\}}$

Forward operator	Algorithm	Remarks
$\boldsymbol{R}_s^n \to \boldsymbol{R}_l^n \to \boldsymbol{P}_s^n \to \boldsymbol{P}_l^n$	FHTR $\to$ iFHTR iSVTR	Hilbert transform; singular functions
$\boldsymbol{R}_s^{\{n,l,c\}}$ $\boldsymbol{R}_s^{\{n,l,c\}} \to \boldsymbol{R}_l^{\{n,l,c\}} \to \boldsymbol{P}_s^n \to \boldsymbol{P}_l^n$ $\boldsymbol{R}_s^{\{n,l,c\}} \to \boldsymbol{R}_l^{\{n,l,c\}} \to \boldsymbol{P}_s^n \to \boldsymbol{P}_l^n$	P-CuReD CLIF $\to$ i-CLIF PFTR $\to$ i-PFTR	Fourier domain based
$\boldsymbol{R}_s^{\{n,l,c\}} \to \boldsymbol{R}_l^{\{n,l,c\}} \to \boldsymbol{P}_s^{\{n,l,c\}} \to \boldsymbol{P}_l^{\{n,l,c\}}$	CGNE SD PKI	iterative algorithms, adjoint operators

Outline

Roof WFS: nonlinear transmission mask model

$$s_x^{\{n,l,c\}} = \boldsymbol{R}_t^{\{n,l,c\}} \phi$$

$$\begin{aligned}
(\boldsymbol{R}_t^{\{n,l,c\}} \phi)(x,y) &= \frac{1}{\pi} \mathcal{X}_{\Omega_y \times \Omega_x}(x,y) \int_{-X(y)}^{X(y)} \frac{\sin\left[\phi(x',y) - \phi(x,y)\right] k_{\{n,l,c\}}(x'-x)}{x-x'} dx' \\
&= \mathcal{X}_{\Omega_y \times \Omega_x}(x,y) \cos(\phi(x,y)) \cdot \left[\mathcal{X}_{\Omega_y \times \Omega_x}(\cdot,y) \sin(\phi(\cdot,y)) * \frac{k_{\{n,l,c\}}(\cdot)\delta(y)}{\pi \cdot} \right] \\
&- \mathcal{X}_{\Omega_y \times \Omega_x}(x,y) \sin(\phi(x,y)) \cdot \left[\mathcal{X}_{\Omega_y \times \Omega_x}(\cdot,y) \cos(\phi(\cdot,y)) * \frac{k_{\{n,l,c\}}(\cdot)\delta(y)}{\pi \cdot} \right]
\end{aligned}$$

Inversion: Nonlinear Landweber method, nonlinear CG, nonlinear SD, ...

$$\phi_{k+1} = \phi_k + \left(\left(\boldsymbol{R}_t^{\{n,l,c\}} \right)' \right)^* \left(s_x - \boldsymbol{R}_t^{\{n,l,c\}} \phi_k \right), \qquad k \in \mathbb{N}$$

Comparison of quality

Algorithm	Quality in end-to-end simulations (OCTOPUS)			
	METIS mod 0	METIS mod 4	XAO mod 0	XAO mod 4
Photon flux	10000 ph/pix/it	10000 ph/pix/it	50 ph/pix/it	50 ph/pix/it
Frame rate	1kHz	1kHz	3kHz	3kHz
Matrix-Vector Multiplication (MVM)	$\approx$ 0.62 [1] (1000ph)	0.80 [2] (1000ph) 0.89 [3]		0.96 0.96
Preprocessed CuReD (P-CuReD)		0.89	0.91	0.96
Conv. with Linearized Inverse Filter (CLIF)			0.88	0.94
Pyramid FTR (PFTR)			0.88	0.94
Finite Hilbert Transform Rec. (FHTR)			0.85	
Singular Value Type Reconstructor (SVTR)	0.77		0.88	
Conjugate Gradient for Normal Eq. (CGNE)				
Steepest Descent (SD)	$\geq$ 0.79		0.90	
Pyramid Kaczmarz Iteration (PKI)	0.81		0.92	
Nonlinear Landweber (NL)	$\geq$ 0.83			

[1] M. Le Louarn et. al., Latest AO simulation results for the E-ELT, poster AO4ELT5.

[2] Results provided by ESO.

[3] MMSE reconstructor in YAO, results provided by Stefan Hippler.

Comparison of computational load

Algorithm	Modulation no	Modulation small	Modulation large	Complexity	Remarks
Matrix-Vector Multiplication (MVM)	+	+	+	$O(n^2)$	baseline;
Fourier Transform Reconstructor (FTR)	–	–	+	$O(n \log n)$	geometrical model
Preprocessed CuReD (P-CuReD)	+	+	+	$O(n)$	Fourier domain based
(i)-Conv. with Linearized Inverse Filter (CLIF)	+	+	+	$O(n^{3/2})$	(iteartive)
(i)-Pyramid FTR (PFTR)	+	+	+	$O(n \log n)$	
Hilbert Transform Reconstructor (HTR)	+	–	–	$O(n \log n)$	(iterative) inversion of finite
(i)-Finite Hilbert Transform Rec. (FHTR)	+	–	–	$O(n^{3/2})$	Hilbert transform;
(i)-Singular Value Type Reconstructor (SVTR)	+	–	–	$O(n^{3/2})$	singular functions
Steepest Descent (SD)	+	+	+	$O(n^{3/2})$	adjoint operators
Pyramid Kaczmarz Iteration (PKI)	+	+	+	$O(n^{3/2})$	
Nonlinear Landweber (NL)	+	+	+	$O(n^{3/2})$	Fréchet derivative
					its adjoint

Non-linear models

Model	Roof-WFS	Pyramid WFS
Nonlinear transmission mask	$\boldsymbol{R}_t^{\{n,l,c\}}$	$\boldsymbol{P}_t^{\{n,l,c\}}$
Nonlinear phase mask without interference	$\boldsymbol{R}_p^{\{n,l,c\}}$	$\boldsymbol{P}_p^{\{n,l,c\}}$
Nonlinear phase mask withi interference	$\boldsymbol{R}_i^{\{n,l,c\}}$	$\boldsymbol{P}_i^{\{n,l,c\}}$

Pyramid WFS: nonlinear transmission mask model

$$(\boldsymbol{P}_t^{\{n,l,c\}}\phi)(x,y) = (\boldsymbol{R}_t^{\{n,l,c\}}\phi)(x,y) - \frac{1}{\pi^3}\int_{-X(y)}^{X(y)}\int_{-Y(x)}^{Y(x)}\int_{-Y(x)}^{Y(x)} \frac{\sin[\phi(x',y')-\phi(x,y'')]p_{\{n,c\}}(x'-x,y'-y'')}{(x-x')(y-y')(y-y'')}\,dy''\,dy'\,dx'$$

Inversion: nonlinear Landweber, nonlinear CG, nonlinear SD, ...

$$\psi_{det}(x,y) = \frac{1}{2\pi}\left(\psi_{aper} * \mathcal{F}^{-1}\{OTF_{pyr}^{t}\}\right)(x,y)$$

$$OTF_{pyr}^{t}(\xi,\eta) = \sum_{m=0}^{1}\sum_{n=0}^{1} T^{mn}(\xi,\eta)$$

$$T^{mn}(\xi,\eta) = H_{2d}\left[(-1)^m\cdot\xi,(-1)^n\cdot\eta\right]$$

$$I(x,y) \approx \sum_{n=0}^{1}\sum_{m=0}^{1}\psi_{n,m}(x,y)\cdot\overline{\psi_{n,m}(x,y)}$$

Roof & Pyramid WFS: nonlinear phase mask model w/o interference

Nonlinear, phase mask, no interference:

$$\psi_{det}(x,y) = \frac{1}{2\pi}\left(\psi_{aper} * \mathcal{F}^{-1}\{OTF^p_{pyr}\}\right)(x,y)$$

$$OTF^p_{pyr}(\xi,\eta) = \exp(-i\cdot\Pi(\xi,\eta))\cdot OTF^t_{pyr}(\xi,\eta)$$

$$I(x,y) \approx \sum_{n=0}^{1}\sum_{m=0}^{1}\psi_{n,m}(x,y)\cdot\overline{\psi_{n,m}(x,y)}$$

Nonlinear, phase mask, with interference:

$$\psi_{det}(x,y) = \frac{1}{2\pi}\left(\psi_{aper} * \mathcal{F}^{-1}\{OTF^p_{pyr}\}\right)(x,y)$$

$$OTF^p_{pyr}(\xi,\eta) = \exp(-i\cdot\Pi(\xi,\eta))\cdot OTF^t_{pyr}(\xi,\eta)$$

$$\begin{aligned} I(x,y) &= \sum_{n=0}^{1}\sum_{m=0}^{1}\psi_{n,m}(x,y)\cdot\overline{\psi_{n,m}(x,y)} \\ &+ 2\sum_{n=0}^{1}\sum_{m=0}^{1}\sum_{n'=0,n'\neq n}^{1}\sum_{m'=0,m\neq m}^{1} Re[\psi_{n,m}(x,y)\cdot\overline{\psi_{n',m'}(x,y)}]\end{aligned}$$

Extenstion of algorithms to other non-linear models

Model	Roof-WFS	Pyramid WFS
Nonlinear transmission mask	$\boldsymbol{R}_t^{\{n,l,c\}}$	$\boldsymbol{P}_t^{\{n,l,c\}}$
Nonlinear phase mask without interference	$\boldsymbol{R}_p^{\{n,l,c\}}$	$\boldsymbol{P}_p^{\{n,l,c\}}$
Nonlinear phase mask withi interference	$\boldsymbol{R}_i^{\{n,l,c\}}$	$\boldsymbol{P}_i^{\{n,l,c\}}$

Forward operator	Algorithm	Remarks
$\boldsymbol{R}_t^{\{n,l,c\}} \to \boldsymbol{R}_p^{\{n,l,c\}} \to \boldsymbol{R}_i^{\{n,l,c\}}$	NL, nCG, ...	Fréchet derivative,
$\boldsymbol{P}_t^{\{n,l,c\}} \to \boldsymbol{P}_p^{\{n,l,c\}} \to \boldsymbol{P}_i^{\{n,l,c\}}$	NL, nCG, ...	adjoint

Outline

Summary

- Roof $\longrightarrow$ pyramid
- Linearized models $\longrightarrow$ non-linear models
- A wide spectrum of algorithms developed and studied: linear and non-linear
- Quality and speed better than MVM !
- Can handle spiders with a linear method !
- Open questions: best reconstruction quality model, ncpa, deeper understanding of spiders
- Go on-sky...

Urban Bitenc et al., On-sky tests of the CuReD and HWR fast wavefront reconstruction algorithms with CANARY. Monthly Notices of the Royal Astronomical Society 448(2), 1199-1205 (2015).

Thanks

Thank you for attention!