What happens when nuclear fuel is OVER?

$$\frac{dP}{dr} < -\frac{G\,m(r)}{r^2}\,\rho$$

hydrostatic equilibrium is broken and STAR CONTRACTS

Can the star contract forever (ie $R \to \emptyset$)? NO because

Let us calculate the $\boxed{\text{potential energy}}$ of the star

$$W = -\int_0^M \frac{G\,m(r)}{r}\,dm(r) \underset{\uparrow}{=} -\int_0^R \frac{G\,m(r)}{r}\,\rho\,4\pi r^2\,dr =$$

using continuity equation

$$\underset{\uparrow}{=} \int_0^R \frac{dP}{dr}\,4\pi r^3\,dr \underset{\uparrow}{=} \int_0^R \frac{d}{dr}\left(P\,4\pi r^3\right)dr - \int_0^R P\,4\pi\,3 r^2\,dr$$

using hydrostatic equilibrium — integrate by parts

$$= P\,4\pi r^3\Big|_0^R - 3\int_0^R 4\pi P r^2\,dr = \cancel{\underset{\emptyset}{P(R)}\,4\pi R^3 - P(0)\,4\pi\,\emptyset^3} +$$

$$-3\int_0^R 4\pi P r^2\,dr$$

$$\Rightarrow \quad W = -3\int_0^R 4\pi P r^2\,dr$$

If $P = \dfrac{\rho}{\mu m_p} k_B T$ (ideal gas, no radiation pressure, only gas pressure)

$$W = -3 \int_0^R 4\pi P r^2 \, dr = -3 \int_0^R 4\pi \frac{\rho}{\mu m_p} k_B T \, r^2 \, dr$$

$$\sim -\frac{3 k_B \bar{T}}{\mu m_p} \int_0^R 4\pi r^2 \rho \, dr$$

$$\sim -\frac{3 k_B \bar{T}}{\mu m_p} M$$

But $W = -\dfrac{G M^2}{R}$

$$\Rightarrow -\frac{G M^{\cancel{2}}}{R} \sim -\frac{3 k_B \bar{T}}{\mu m_p} \cancel{M}$$

$$\Rightarrow \bar{T} \propto \frac{\mu m_p}{3 k_B} \frac{G M}{R}$$

$$\boxed{\bar{T} \propto \frac{M}{R}}$$

if star contracts $\bar{T}$ rises

$$\boxed{\rho \propto \frac{M}{R^3}}$$

if star contracts ρ rises much faster than $\bar{T}$

$\Longrightarrow$ At some point density increases so much that electron gas becomes DEGENERATE:

1) Typical momentum difference between electrons in a Maxwell-Boltzmann gas

$$\Delta p_e \sim \left(6\, \underbrace{m_e}_{\text{electron mass}}\, k_B \bar{T}\right)^{1/2}$$

$$\sim \left(2\, m_e\, \mu\, m_p \frac{G M}{R}\right)^{1/2}$$

using $k_B \bar{T} \sim \frac{\mu\, m_p}{3} \frac{G M}{R}$

2) Typical separation between electrons

$$\Delta q_e \sim \left(\frac{\mu_e\, m_p}{\rho}\right)^{1/3} \sim \left(\frac{\mu_e\, m_p\, R^3}{M}\right)^{1/3}$$

Volume occupied by electrons in phase space:

$$\left(\Delta p_e \ \Delta q_e\right)^3 \sim \left(\mu m_p \ \frac{m_e G M}{R}\right)^{3/2} \ \frac{\mu_e m_p \ R^3}{M}$$

$$\sim \mu^{3/2} \ m_p^{5/2} \ \mu_e \ m_e^{3/2} \ G^{3/2} \ M^{1/2} \ R^{3/2}$$

Expressing constants:

$$\sim 180 \ h^3 \ \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^{3/2}$$

When $R \sim 3 \times 10^{-2} R_\odot$, $\left(\Delta p_e \Delta q_e\right)^3 \sim h^3$

$\Rightarrow$ Pauli exclusion principle becomes important and Fermi-Dirac statistics must be used

in other words

$\boxed{\text{ELECTRON DEGENERACY PRESSURE SUPPORTS A STAR}}$

even if temperature does not rise

Is there a MAXIMUM MASS for electron degeneracy pressure to support a star?

CHANDRASEKHAR (1931) MASS

Let's derive it with LANDAU (1932) ARGUMENT

N = number of Fermions in a star of radius R

$n \propto \frac{N}{R^3}$ (number density of fermions)

According to Pauli exclusion principle, the volume per Fermion is

$$V \sim \frac{1}{n} \longrightarrow \Delta x \sim \left(\frac{1}{n}\right)^{1/3}$$

From Heisenberg uncertainty principle ($\Delta x \, \Delta p \sim \hbar$)

the momentum of Fermions is then

$$\boxed{\begin{aligned} \Delta p &\sim \hbar \, n^{1/3} \\ &\sim \hbar \frac{N^{1/3}}{R} \end{aligned}}$$

Thus, the Fermi energy of a gas particle is

$$E_F \sim \hbar n^{1/3} c \sim \hbar c \frac{N^{1/3}}{R}$$

The gravitational energy per baryon is

$$E_G = -\frac{G M m_B}{R} = -\frac{G N m_B^2}{R}$$

pressure comes from electrons
but mass is dominated by baryons

Total energy E :

$$E = E_F + E_G = \hbar c \frac{N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

Equilibrium is achieved only for a minimum of E

- If E is positive (ie small N), E can be decreased by increasing R (E_F decreases)
  $\Rightarrow$ we find a stable equilibrium for finite R
- If E is negative (ie large N), $E \to -\infty$ if $R \to 0$
  NO EQUILIBRIUM is possible, star collapses

The maximum baryon number for equilibrium is thus obtained if $0 = E = E_F + E_G$

$$N = \left(\frac{\hbar c}{G m_B^2}\right)^{3/2} \sim 2 \times 10^{57}$$

Thus the maximum total mass which can be at equilibrium is

$$\boxed{M_{max} = N m_B \sim 1.5 \, M_\odot}$$

Chandrasekhar mass (should be $1.4 \, M_\odot$ with more accurate calculation)

No equilibrium by electron degeneracy pressure exists ABOVE this mass

Stars with mass $< M_{max}$ are called WHITE DWARFS

WHAT HAPPENS ABOVE CHANDRASEKHAR MASS?

Above Chandrasekhar mass, a star can still be supported by NEUTRON degeneracy pressure

A star supported by neutron degeneracy pressure is called NEUTRON STAR

We know the hydrostatic equilibrium equation

$$\frac{dP}{dr} = - \frac{G \, m(r)}{r^2} \rho(r)$$

If gas is RELATIVISTIC, Oppenheimer & Volkoff (1939) show that the hydrostatic equilibrium equation becomes

$$\boxed{\frac{dP}{dr} = -\frac{G}{r^2}\left(\rho(r) + \frac{P(r)}{c^2}\right)\left(m(r) + \frac{4\pi r^3 P(r)}{c^2}\right)\left(1 - \frac{2Gm(r)}{c^2 r}\right)^{-1}}$$

if $\frac{P(r)}{c^2} << 1$, $\frac{Gm(r)}{r\,c^2} << 1 \Rightarrow \frac{dP}{dr} \to -\frac{Gm(r)}{r^2}\rho(r)$

Solving Oppenheimer-Volkoff (OV) equation and imposing that the central pressure is $< \infty$ gives a MAXIMUM COMPACTNESS

$$\boxed{\frac{GM}{c^2 R} < \frac{4}{9}}$$

Stars with $M > \frac{4}{9} R \frac{c^2}{G}$ cannot be supported by relativistic pressure

For example if $R = 10\,km$

$$\boxed{M_{max} = \frac{4}{9}\frac{R\,c^2}{G} \simeq 3\,M_\odot \left(\frac{R}{10\,km}\right)}$$

Solving it for a PURE IDEAL NEUTRON GAS, the OV equation predicts a maximum neutron star mass

$$\left\{\begin{array}{l} M^{NS}_{max} = 0.7\,M_\odot \\ R = 9.6\,Km \\ \rho_c = 5\times10^{15}\,g/cm^3 \end{array}\right.$$

For more realistic equations of state

$$M^{NS}_{max} \sim 2 - 3\,M_\odot$$

Above this mass NO known pressure source can balance gravity in the OV equation

$\Longrightarrow$ $\boxed{\text{BLACK HOLES}}$