QUICK SUMMARY of STAR EVOLUTION

- When I talk about stellar luminosity

$$L = 4\pi R^2 \sigma T_{eff}^4$$

R: radius

σ: Stefan-Boltzmann constant $5.67 \times 10^{-5} \frac{erg}{cm^2 K^4 s}$

T_{eff}: effective temperature

T_{eff} = temperature of a black body with same luminosity per surface area

- COLOUR of a star depends on the wavelength at which the star has maximum intensity (WIEN LAW)

$$\lambda_{max} T_{eff} \simeq 2.9 \times 10^6 \text{ nm K}$$

- Stars are in LOCAL THERMODYNAMIC EQUILIBRIUM (LTE)

mean free path of gas particles in a star =

$$\left(\frac{V}{N}\right)^{1/3} = \left(\frac{V}{M/m_p}\right)^{1/3} \underset{\text{for the Sun}}{\sim} \left(\frac{4}{3}\pi R_\odot^3 \frac{m_p}{M_\odot}\right)^{1/3}$$

$$\sim 10^{-8} \text{ cm}$$

$R_\odot = 6.95 \times 10^{10}$ cm

$M_\odot = 1.989 \times 10^{33}$ g

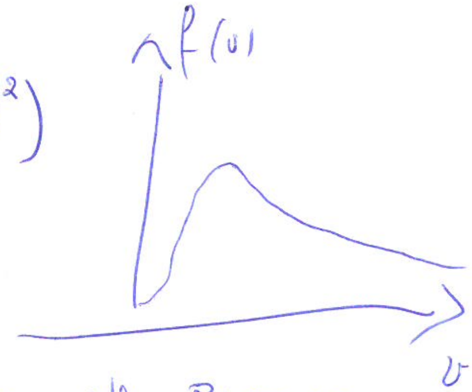
$m_p = 1.67 \times 10^{-24}$ g

⇒ collisions between gas particles lead to RELAXATION in very short time

⇒ particles reach LOCAL ENERGY EQUILIBRIUM

⇒ velocity distribution at energy equilibrium is defined by MAXWELL - BOLTZMANN distribution

$$f(v) \propto v^2 \exp(-v^2)$$



Gas particles are in EQUILIBRIUM with PHOTONS because of the frequent collisions

⇒ Also photons reach LOCAL THERMODYNAMIC EQUILIBRIUM (LTE):

temperature of gas and photons is the same

⇒ Distribution of photons is described by the

BLACK BODY RADIATION

energy distribution of photons described by the $\boxed{\text{PLANCK CURVE}}$

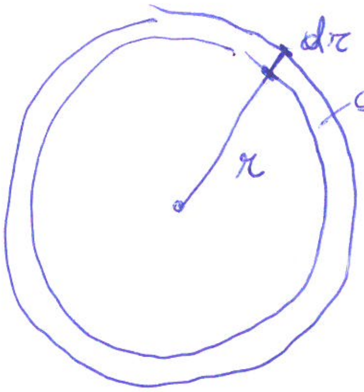
radiance of a body

$$\vec{B}_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/k_BT} - 1\right)^{-1}$$

$h = 6.626 \times 10^{-27}$ erg s

$k_B = 1.38 \times 10^{-16}$ erg/K

- Stars satisfy CONTINUITY EQUATION (MASS CONSERVATION)



<u>Volume element</u>

$$dV = 4\pi r^2 dr$$

Mass

$$dm = \rho \, dV$$

$$\Rightarrow \boxed{\frac{dm}{dr} = 4\pi r^2 \rho}$$

$$M = \int_0^M dm = \int_0^R 4\pi r^2 \rho \, dr$$

- In thermal equilibrium, the HEAT flowing through a spherical surface is

$$dl = 4\pi r^2 q \rho \, dr$$

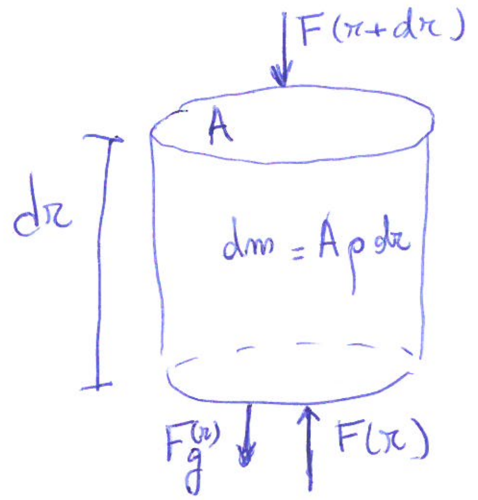
q = rate of energy generation per unit mass $\frac{dE}{dt\,dm}$

$\Rightarrow$ <u>Total luminosity of a star</u>

$$L = \int_0^M dl = \int_0^M q \, dm = \int_0^R 4\pi r^2 q \rho \, dr$$

• All stars are in <u>HYDROSTATIC EQUILIBRIUM</u> for most of their life (only exceptions: some variable stars and collapse before SN)

Let us consider a mass element:



$F(r+dr) = P(r+dr)\ A$

force exerted by GAS PRESSURE in $r+dr$

$F(r) = P(r)\ A$

force exerted by gas pressure in r

$F_g(r) = -\dfrac{G\,m(r)}{r^2}\,dm$

gravity force

Total force:

$$F_{TOT}(r) = F_g(r) + F(r+dr) + F(r)$$

$$= -\frac{G\,m(r)\,dm}{r^2} + \underbrace{\left[-P(r+dr) + P(r)\right]}_{-dP} A$$

Stars do not collapse neither expand $\Rightarrow$

$$F_{TOT}(r) = \emptyset$$

$$\Rightarrow \underline{dP}\, A = -\frac{G\, m(r)}{r^2}\, dm$$

We use the fact that

$$A\, dr = V, \quad dm = A \rho\, dr$$

and divide both terms by V

$$\frac{A}{V}\, dP = -\frac{G\, m(r)}{r^2} \frac{dm}{V}$$

$$\frac{\cancel{A}}{\cancel{A}} \frac{dP}{dr} = -\frac{G\, m(r)}{r^2} \frac{\cancel{A} \rho\, \cancel{dr}}{\cancel{A}\, \cancel{dr}}$$

$$\Rightarrow \boxed{\frac{dP}{dr} = -\frac{G\, m(r)}{r^2} \rho}$$

equation of <u>hydrostatic equilibrium</u>

We did not say any thing about the origin of P

- If we assume a star is made of IDEAL GAS then P = gas pressure given by EoS

$$P_{gas} = \frac{\mathcal{R}}{\mu} \rho T$$

$\mathcal{R}$: $8.3 \times 10^7 \frac{erg}{K\, mol}$ (specific ideal gas constant)

μ: mean molecular weight

- Pressure may also be provided by RADIATION PRESSURE

$$P_{rad} = \frac{1}{3} \int_0^\infty \underbrace{h\nu}_{\text{energy carried by single photon}} \underbrace{n(\nu)}_{\text{photon density}} d\nu$$

$h = 6.6 \times 10^{-27}$ erg.s

units of

$$\frac{E}{V} = \frac{F\, \cancel{Length}}{A\, \cancel{Length}}$$

Note:

$h\nu\, n(\nu) = \underbrace{u_\nu}_{\text{energy density}}$, follows PLANCK function for black body radiation

$$\Rightarrow P_{rad} = \int_0^\infty u_\nu\, d\nu = \frac{1}{3} a T^4 = \frac{1}{3} \frac{8\pi^5 K_B^4}{15 c^3 h^3} T^4$$

So the total pressure in a star is given by

$$P = P_{gas} + P_{rad} = \frac{\mathcal{R}}{\mu}\rho T + \frac{1}{3} a T^4$$

different dependence on T

if T high ⇒ rad pressure dominates

- PHOTON TRANSFER:

We said nothing about how photons travel through the star

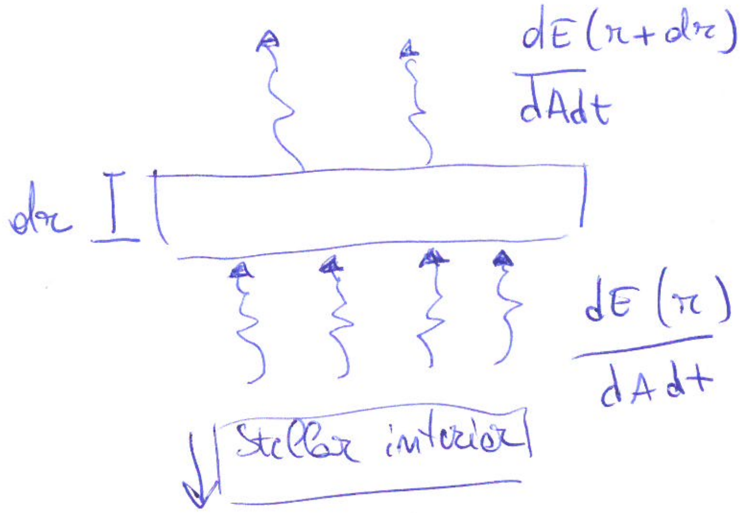
Photons have extremely small free path

⇒ the same photon cannot travel through the entire star

⇝ photons are continuously absorbed and re-emitted

How?

Consider a slab



Energy absorbed in the slab per unit area and time

$$\frac{dE(r)}{dA\,dt} - \frac{dE}{dA\,dt}(r+dr) \propto \underbrace{\frac{dE(r)}{dA\,dt}}_{\text{available energy}} \ \underbrace{\rho}_{\text{density of absorbers}} \ \underbrace{dr}_{\text{thickness of absorbing layer}}$$

$$\frac{g\,cm^2}{s^2\,cm^2\,s} \sim \frac{g}{s^3}$$

$$= -\kappa\,\rho\,\frac{dE(r)}{dA\,dt}\,dr$$

$$\frac{g}{cm^3}\ \frac{g\,cm^2}{cm^2\,s^2\,s}\ cm$$

$\Rightarrow$ κ has units of cm^2/g

$\boxed{\text{OPACITY}}$ of the slab

Mean free path of photons $\propto \dfrac{1}{\kappa\rho}$

Momentum absorbed by the slab (per unit area)

$$\underbrace{\frac{1}{c}\left[\frac{dE(r)}{dA\,dt} - \frac{dE(r+dr)}{dA\,dt}\right]}_{\text{think about it as a luminosity per unit area}} = -\varkappa\rho\frac{dE(r)}{dA\,dt}\frac{1}{c}\,dr$$

$$\frac{\cancel{s}}{cm}\ \frac{g\,\cancel{cm^2}}{s^2\,cm^2\,\cancel{s}} = g\,cm^{-1}\,s^{-2}$$

$$\boxed{L/c \text{ per unit mass} \quad \frac{g\,cm^{\cancel{2}}\,\cancel{s}}{s^{\cancel{3}2}\,\cancel{cm}} = g\frac{cm}{s^2}}$$

It is a pressure:

$$dP = -\varkappa\rho\frac{dE(r)}{dA\,dt}\frac{1}{c}\,dr$$

$$\frac{F}{A} = \frac{g\,cm}{s^2\,cm^2} = \frac{g}{s^2\,cm}$$

$$\frac{dP}{dr} = -\varkappa\rho\frac{dE(r)}{dA\,dt}\frac{1}{c}$$

$\boxed{\text{variation of photon pressure along the slab}}$

Now we express it for $\frac{dE}{dA\,dt}$:

for radiation pressure dominated star $P = \frac{1}{3}aT^4$

$$\frac{dE(r)}{dA\,dt} = -\frac{1}{\varkappa\rho}c\frac{dP}{dr} \overset{\downarrow}{=} -\frac{c}{\varkappa\rho}\frac{d}{dr}\left(\frac{1}{3}aT^4\right)$$

$$\frac{dE}{dA\,dt}(r) = -\frac{c}{\varkappa\rho}\,\frac{1}{3}a\,4T^3\,\frac{dT}{dr}$$

But [from $\oint$] $\frac{dE}{dt} = \ell \quad dA = 4\pi r^2 \rightarrow \frac{dE}{dA\,dt} = \frac{\ell}{4\pi r^2}$

$$\Rightarrow \boxed{\frac{dT}{dr} = -\frac{\ell}{4\pi r^2}\,\frac{\varkappa\rho}{c}\,\frac{3}{4a}\,T^{-3}}$$

Tells us how temperature changes as function of radius, luminosity, density

We can now derive the <u>MASS-LUMINOSITY RELATION</u> of a star, at least for radiation pressure dominated stars

Let us assume for simplicity

$$\rho \sim \frac{M}{\frac{4}{3}\pi R^3} \qquad \frac{dT}{dr} \sim \frac{T}{R}$$

$$\frac{T}{R} \sim \frac{dT}{dr} = -\frac{L}{4\pi R^2}\frac{\varkappa\rho}{c}\frac{3}{4a}T^{-3}$$

$$\frac{T}{\cancel{R}} \sim \frac{L}{R^{\cancel{2}}}\frac{M}{R^3}T^{-3}$$

$$\Rightarrow \boxed{L \sim \frac{T^4}{M}R^4}$$

Ideal gas law $T \propto \frac{\mu}{\mathcal{R}}\frac{P}{\rho} \sim \frac{\mu}{\mathcal{R}}\frac{GM\cancel{\rho}}{R}\frac{1}{\cancel{\rho}}$

using hydrostatic equilibrium

$$P/R \sim \frac{GM\rho}{R^2}$$

$$\Rightarrow L \sim \left(\frac{M}{R}\right)^4 \frac{R^4}{M} \Rightarrow \boxed{L \sim M^3}$$

- hydrostatic equilibrium sets the <u>pressure</u>

$$\frac{P}{R} \propto \frac{GM}{R^2}\rho$$

- radiation equilibrium sets the <u>temperature</u>

$$\frac{T^4}{R} \propto \frac{LM}{R^5}$$

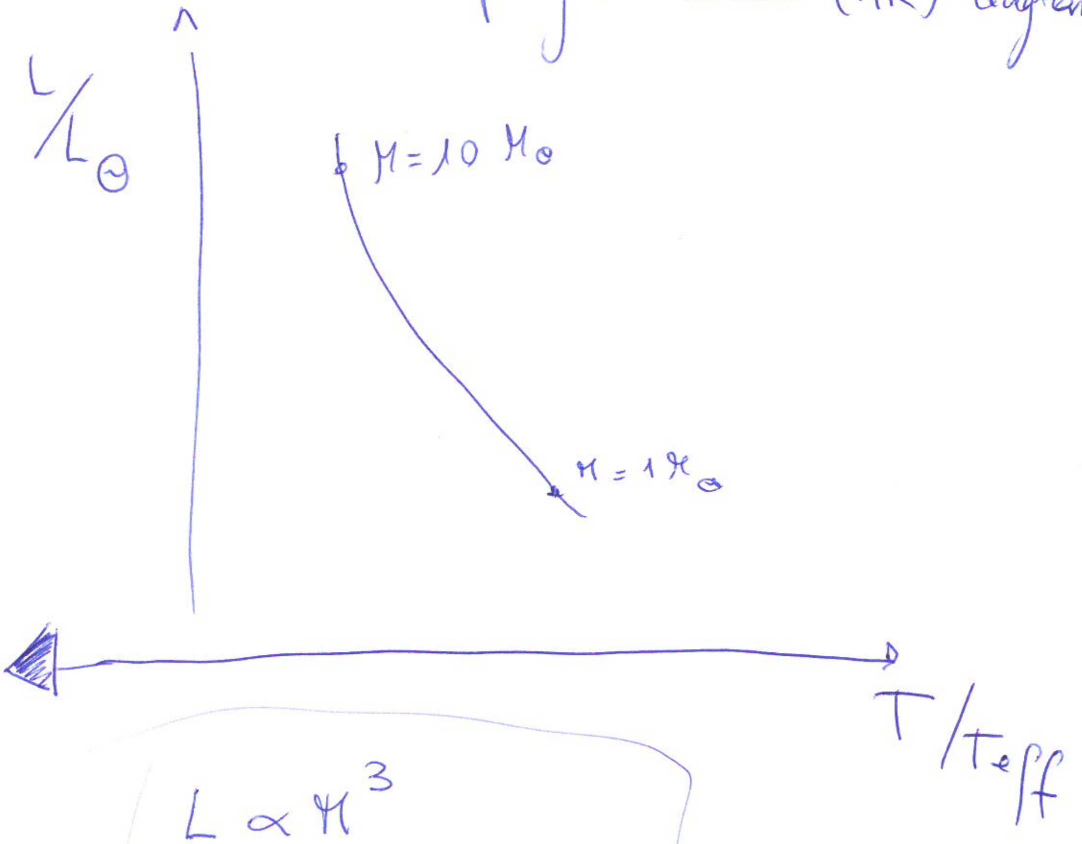
- ideal gas EOS couples the two

$$T \propto \frac{P}{\rho}$$

$$\left[\frac{LM}{R^4}\right]^{1/4} \propto \frac{GM}{R}$$

$$\boxed{L \propto M^3}$$

Hertzsprung - Russel (HR) diagram



$$\boxed{\begin{array}{l} L \propto M^3 \\ L \propto R^2 T_{eff}^4 \end{array}}$$

Combines all information about a star

EDDINGTON LIMIT

What is the maximum luminosity of a star?

(1) relation between luminosity and pressure from radiation transfer equation

$$\frac{dP}{dr} = -\frac{\kappa \rho}{c} \frac{l}{4\pi r^2}$$

(2) hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{G\,m(r)}{r^2}\rho$$

Radiation pressure and gravity are the same if

$$\frac{G\,m(r)}{\cancel{r^2}}\cancel{\rho} = \frac{\kappa \cancel{\rho}}{c} \frac{l}{4\pi \cancel{r^2}}$$

$$\Rightarrow \boxed{l = \frac{4\pi c}{\kappa} G\, m(r)}$$

Eddington luminosity

maximum luminosity to remain in hydrostatic equilibrium

We define Γ (Eddington ratio) as

$$\Gamma = \frac{L}{L_{edd}} = \frac{L\kappa}{4\pi c G} \quad \frac{1}{M}$$

TIMESCALES

① Dynamic timescale: time for a star to collapse under its own gravity

$$\tau_{DYN} = \frac{R}{v} = \sqrt{\frac{R^3}{GM}} \sim 1/\sqrt{G\rho}$$

circular velocity $= \sqrt{\frac{GM}{R}}$

For the Sun:

$$\tau_{DYN} \simeq 1600\ s \left(\frac{R}{6.95\times10^{10}\ cm}\right)^{3/2} \left(\frac{1.989\times10^{33}\ g}{M}\right)^{1/2}$$

② KELVIN - HELMHOLTZ timescale:

time a star needs to convert all its gravitational energy into luminosity

$$\tau_{KH} = \frac{U}{L} = \frac{1}{2}\frac{GM^2}{R}\frac{1}{L}$$

$$\simeq 20\ \mathrm{Myr}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)\left(\frac{L_\odot}{L}\right)$$

$M_\odot = 1.989 \times 10^{33}\,g$ $\quad R_\odot = 6.95 \times 10^{10}\,cm$

$L_\odot = 3.8 \times 10^{33}\,erg/s$ $\quad 1yr \cong 3.15 \times 10^7\,s$

Sun life is longer than 20 Myr ⇒ tells us that a different source of ENERGY is needed, cannot be gravitational energy

3 NUCLEAR TIMESCALE

Energy source of a star comes from nuclear reactions (mostly in the core) unless star is a degenerate star

$$E_{nucl} = \underbrace{f}_{\text{efficiency of mass energy conversion in a nuclear reaction}} M c^2$$

efficiency of mass energy conversion in a nuclear reaction

$f \sim 0.007$ for proton-proton chain

$$4H \rightarrow He + 2\nu + 2e^+ + \underline{\underline{\gamma}}$$

$$\tau_{nucl} = \frac{E_{nucl}}{L} = f \frac{M c^2}{L}$$

$$\simeq 10^{10}\,yr \left(\frac{f}{0.007}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{L_\odot}{L}\right)$$

$$\sim 10^{10}\,yr \left(\frac{f}{0.007}\right)\left(\frac{M_\odot}{M}\right)^2$$

using $L \propto M^3$

First nuclear fuel = Hydrogen in core
is converted to Helium

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star contracts, T rises to ~ 10^8 K

When H exhausted => He burned in core

When He exhausted
→ Small stars cannot reach T to start C burning → collapse to WD

→ stars > 8-9 $M_\odot$
contract till T rises to ~ 0.5×10^9 K

=> start C burning

After Carbon: Neon burning
Oxygen burning
Silicon burning

After each episode of core burning is concluded star core contracts because no layer supported by photon pressure => contraction makes T rise and eventually a new nuclear reaction becomes possible