Michela Mapelli

N-body techniques for astrophysics:
Lecture 6 – Initial conditions

PhD School in Astrophysics,
University of Padova
November 3-12, 2015

OUTLINE of this lecture:

1 – Random sampling of a distribution

2 – Examples

The initial conditions of an N-body simulations:

- a system described as N particles

- it can be (or not) at equilibrium

- I must assign particles (or cells) positions, velocities, masses

If gas: pressure, temperature, density

If stars in 1:1 simulations: radius, luminosity, temperature, metallicity,....

and so on

HOW CAN I DO THIS?

Distribution functions:

1. I need to know the best distribution function of my particles
(for positions, velocities, masses, etc)

2a. If cells: I sample the distribution function on a regular grid
The number of cells is my resolution

2b. If particles: I have to SAMPLE the distribution function with
Monte Carlo technique (I generate particles RANDOMLY
as to be representative of the distribution function)

SAMPLING TECHNIQUES: Inverse transform sampling

Inverse transform sampling (only for invertible distribution functions):

1- Take a distribution function *f(x)* of the quantity *x* I want to sample

2- Integrate it over the range to obtain the PROBABILITY distribution function (cumulative distribution function normalized to 1)

$$g(x) = norm \int f(x)\, dx$$

Gives probability of a value of *x* to occur!!!

3- Randomly sample the cumulative distribution function between min and max value

4- Invert the function to get x $\quad x = g(x)^{-1}$

REPEAT 3 & 4 as many times as you need to get x for N particles

SEE NUMERICAL RECIPES for more details!

http://www2.units.it/ipl/students_area/imm2/files/Numerical_Recipes.pdf

SAMPLING TECHNIQUES: Inverse transform sampling

Example: ECCENTRICITY *e*

1- $f(e) = 2\,e$

called 'thermal distribution of eccentricity' (common in binary systems close to the Sun)

2- cumulative distribution function

$$g(e) = e^2\_max - e^2\_min$$

3- If e^2_max **= 1 and** e^2_min **= 0, use a random generator to generate a random number between 0 and 1**

$$e^2\_ran = \mathrm{rand}(e^2\_max,\, e^2\_min)$$

4- Invert the function:

$$e\_ran = \mathrm{sqrt}(e^2\_ran)$$

REPEAT 3 & 4 as many times as you need to get *e_ran* for N particles

SAMPLING TECHNIQUES: Inverse transform sampling

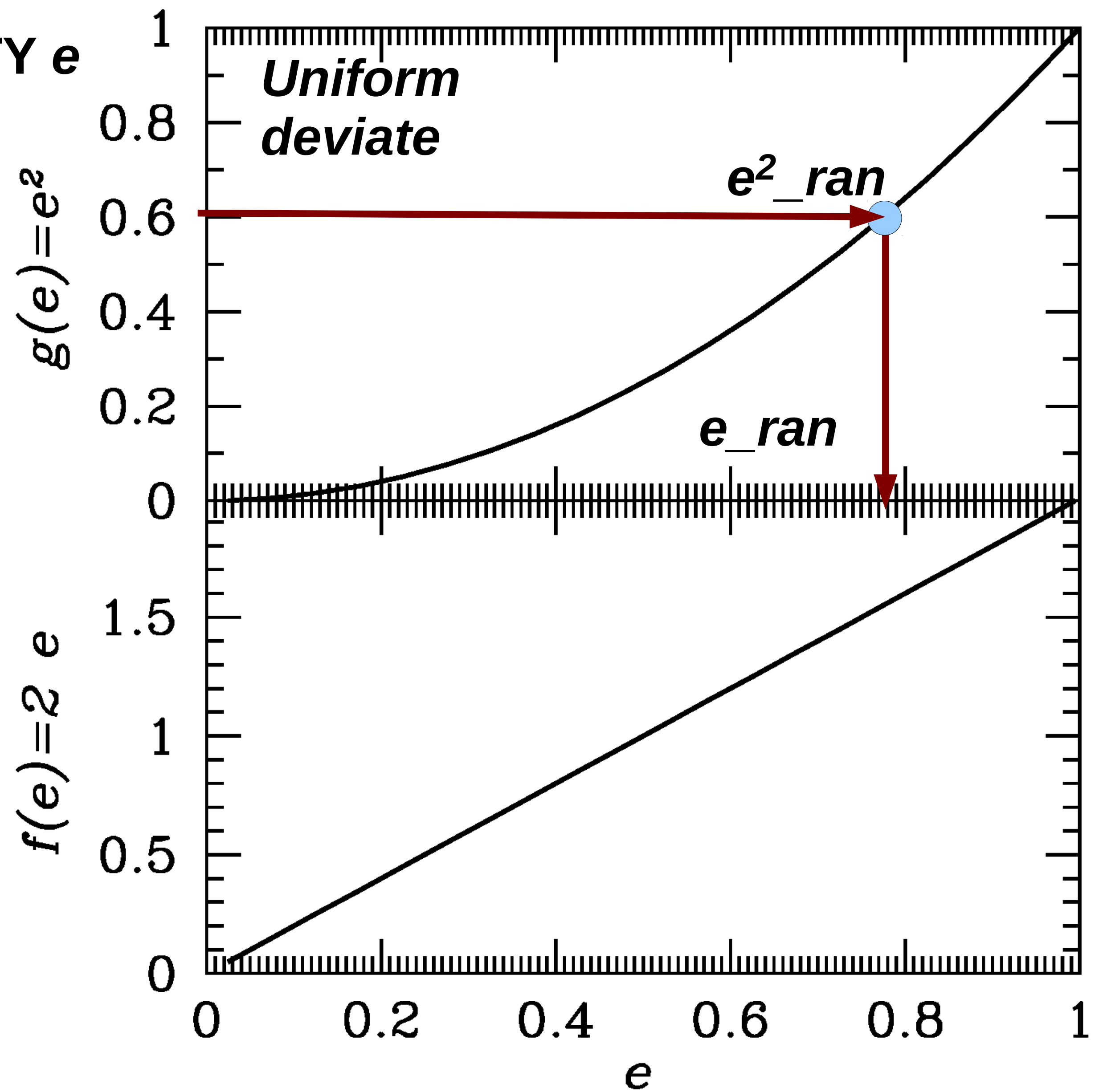
Example: ECCENTRICITY *e*

1- DF
f(e) = 2 e

2- cumulative DF
g(e) = e²

3-
e²_ran* = rand*()

4- Invert the function:
***e_ran* = sqrt(*e²_ran*)**



SAMPLING TECHNIQUES: Inverse transform sampling

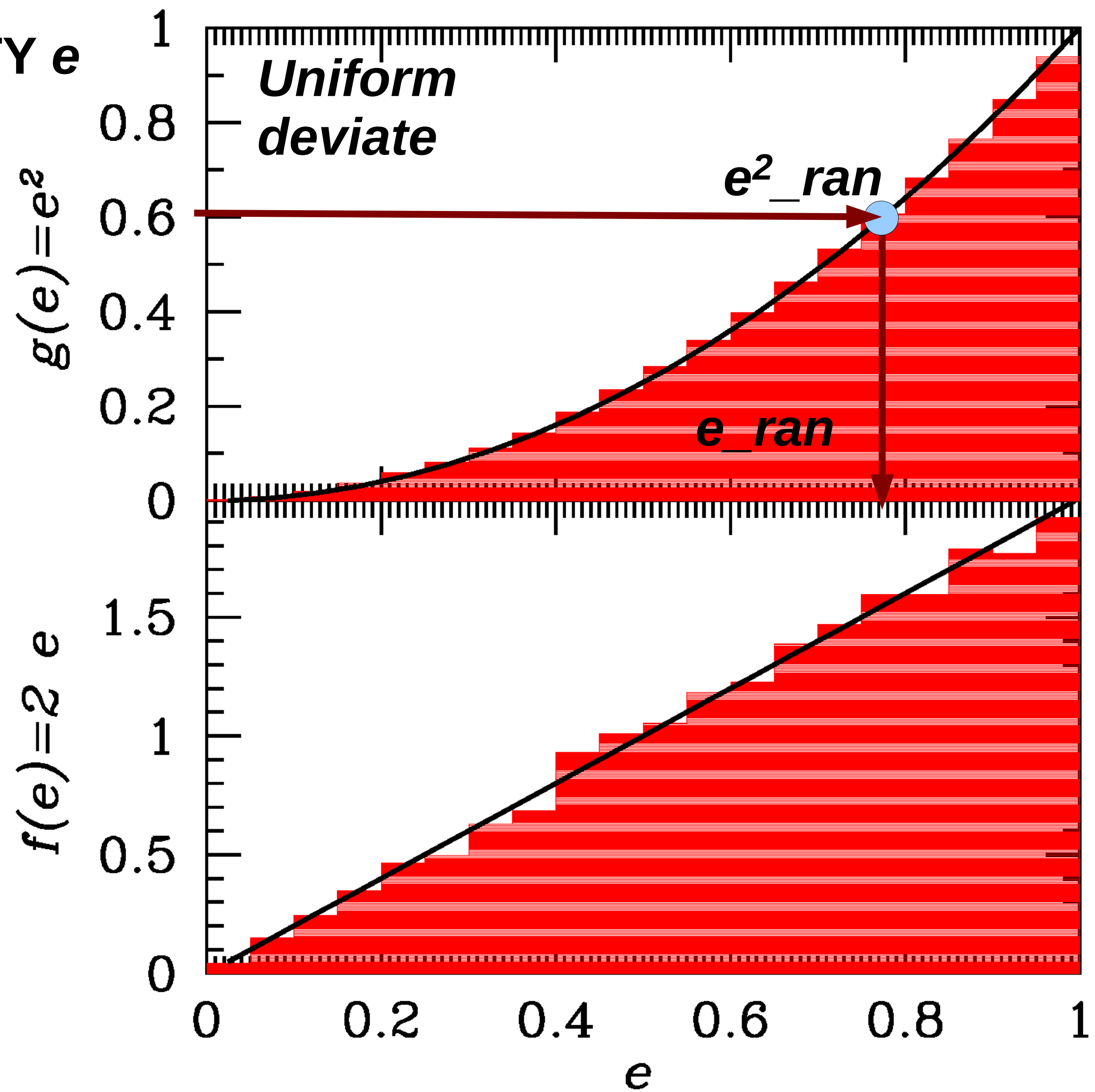
Example: ECCENTRICITY *e*

1- DF
f(e) = 2 e

2- cumulative DF
$g(e) = e^2$

3-
e^2_ran = rand()

4- Invert the function:
$e\_ran = \mathrm{sqrt}(e^2\_ran)$



SAMPLING TECHNIQUES: Inverse transform sampling

AND WHAT DO I DO IF THE CUMULATIVE FUNCTION CANNOT BE INVERTED (EASILY)?

SAMPLING TECHNIQUES: Rejection sampling

Rejection sampling

1- Take a distribution function *p(x)* of the quantity *x* I want to sample

But *p(x)* is difficult/impossible to integrate!

2- Take a second function *f(x) [with f(x)>p(x) everywhere]* that can be easily integrated, to obtain the cumulative distribution function

$$g(x) = norm \int f(x)\, dx$$

Gives probability of a value of *x* to occur!!!

3- Randomly sample *g(x)* between min and max value

4- Invert *y=g(x)* to obtain *x*

x* is distributed according to *f(x)

5- Generate a second random number *m*

Reject *x* if *m>p(x)* and accept *x* if *m<=g(x)*

REPEAT 4 & 5 as many times as you need to get x for N particles

SAMPLING TECHNIQUES: Rejection sampling

Rejection sampling

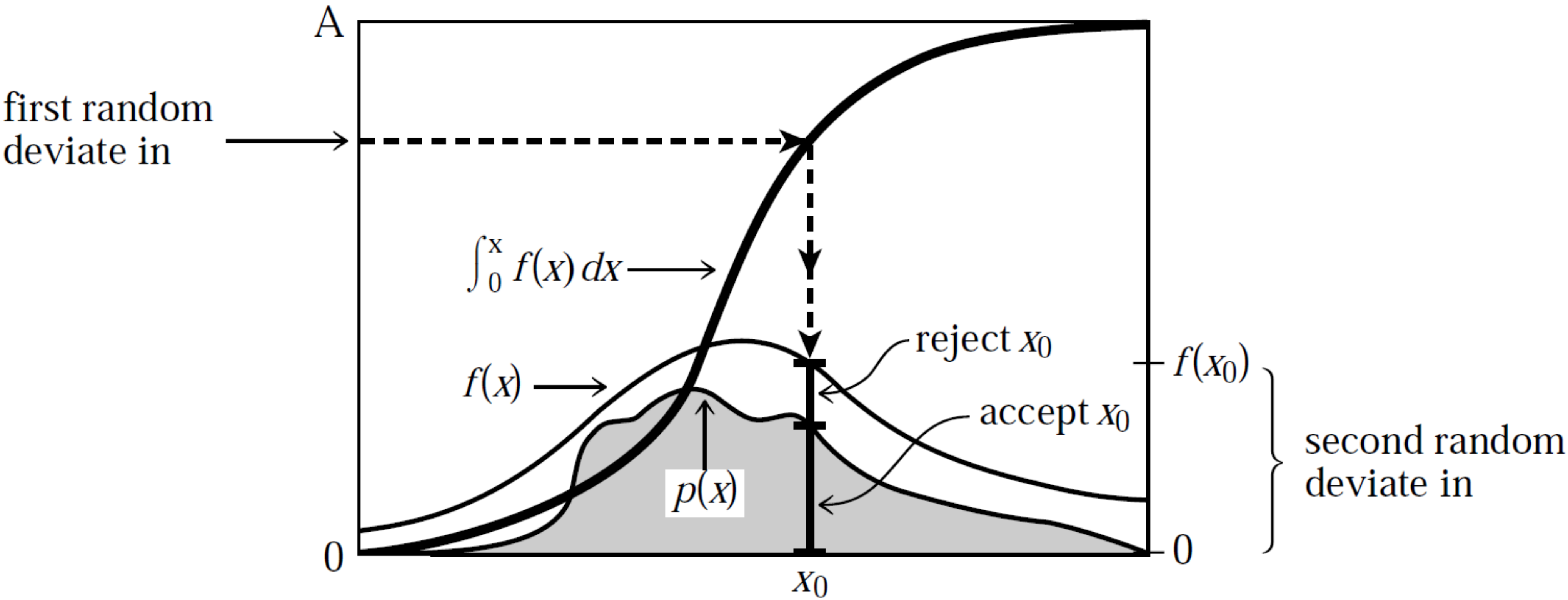


Figure from Numerical recipes in C,

EXAMPLES: example of inverse function sampling

A Hernquist (1990,1993) bulge:

1. distribution function

$$\rho_b(r) = \frac{M_b\, a}{2\,\pi}\,\frac{1}{r\,(a+r)^3}$$

2. cumulative distribution function (mass):

$$M_b(r) = \frac{4\,\pi\,M_b\,a}{2\,\pi}\int_0^\infty \frac{r}{(a+r)^3}dr$$

$$= M_b\,a\,\frac{(a+2\,r)}{(a+r)^2}$$

Normalized to total mass:

$$P(r) = a\,\frac{(a+2\,r)}{(a+r)^2}$$

EXAMPLES: example of inverse function sampling

A Hernquist (1990,1993) bulge:

3. draw uniform random numbers to get *P(r)* $P(r) = a \, \frac{(a + 2\, r)}{(a + r)^2}$

4. invert *P(r)* to get *r* (or use rejection)

5. repeat it for N particles

6. random generate θ (homogeneous in cosθ) and ϕ (homogeneous in 2 π)

7. derive Cartesian coordinates as

$$x = r\ \sin\theta\ \cos\phi$$
$$y = r\ \sin\theta\ \sin\phi$$
$$z = r\ \cos\theta$$

EXAMPLES: example of inverse function sampling

A Hernquist (1990,1993) bulge:

8. velocities are distributed as a Maxwellian with *σ(r)*

$$\sigma^2(r) = \frac{1}{\rho(r)} \int_r^\infty \rho(x) \frac{G\, M_b(x)}{x^2} dx$$

Thus the probability of having a velocity *v* is

$$P(v,r) = 4\,\pi \int_0^v \left(\frac{1}{2\,\pi\,\sigma^2}\right)\, v^2 \exp\left(-\frac{v^2}{2\,\sigma^2}\right) dv$$

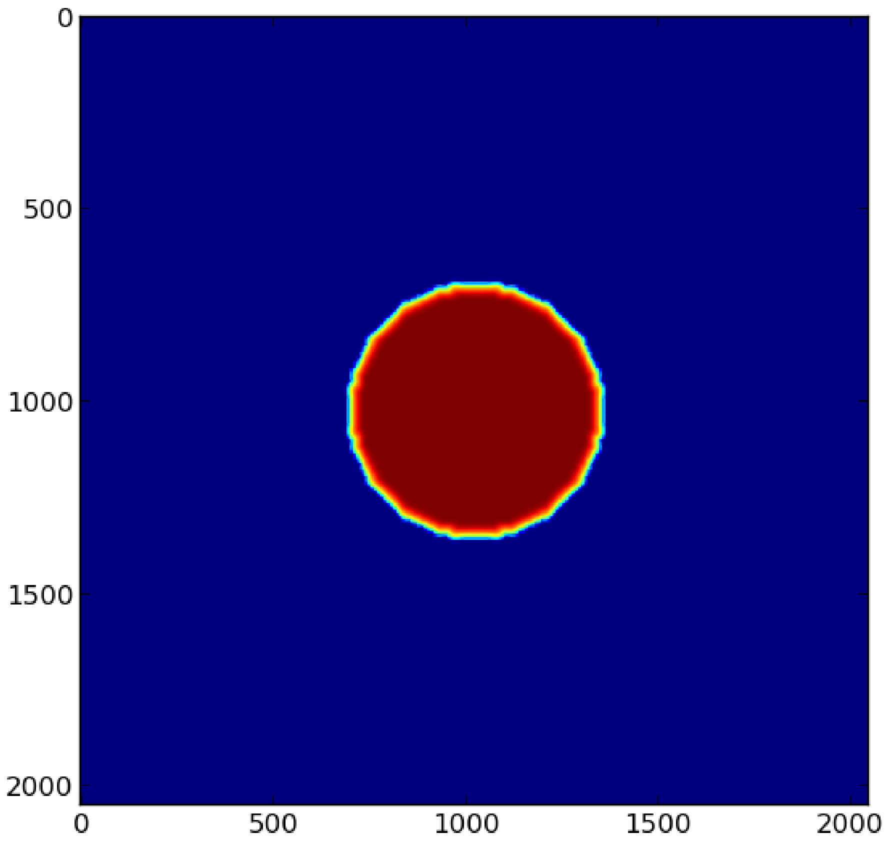
Uniform random numbers are drawn from this distribution and then inverted.

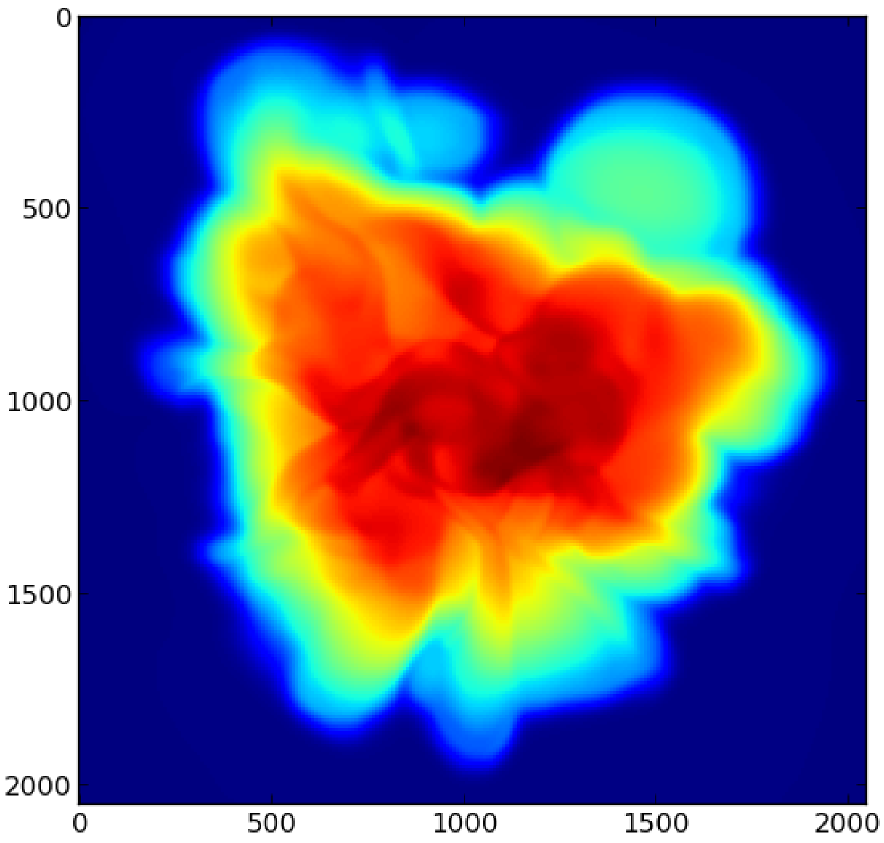
TO GENERATE A MOLECULAR CLOUD:

Uniform sphere of gas;

Total energy = 0 + Gaussian random motions;

Optional: seeded with turbulence as power spectrum.



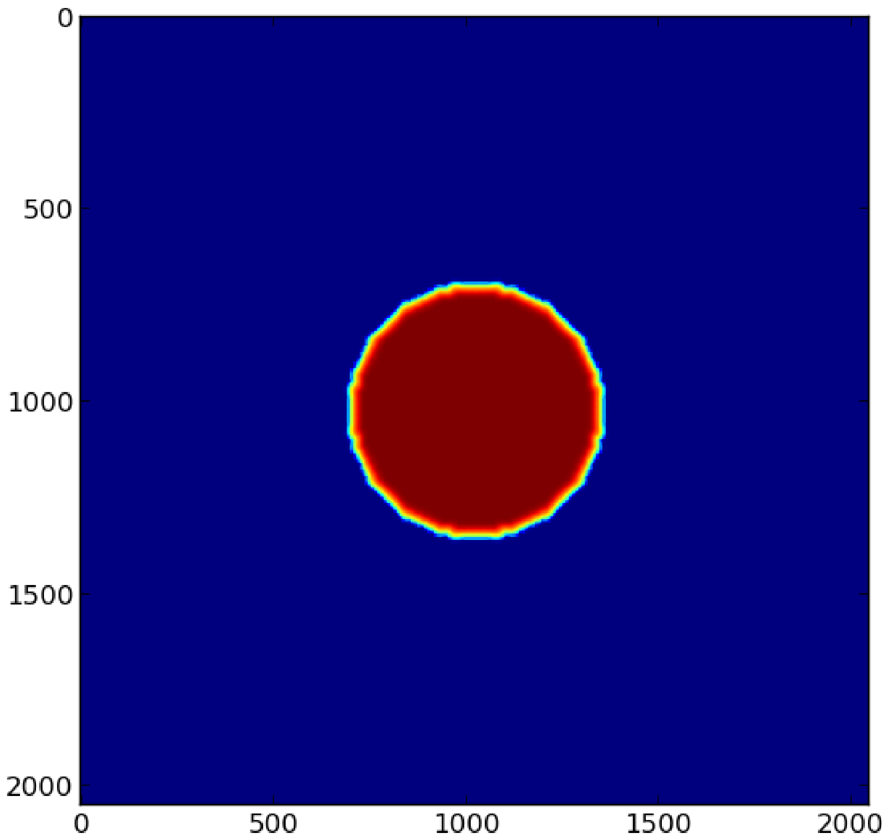


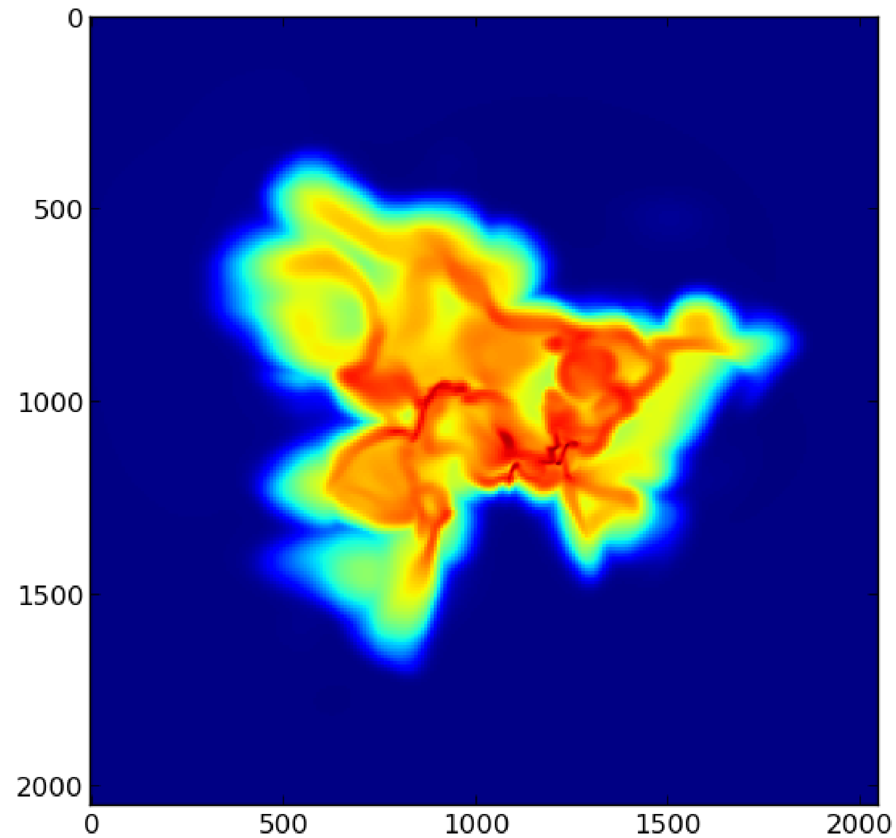
TO GENERATE A MOLECULAR CLOUD:

Uniform sphere of gas;

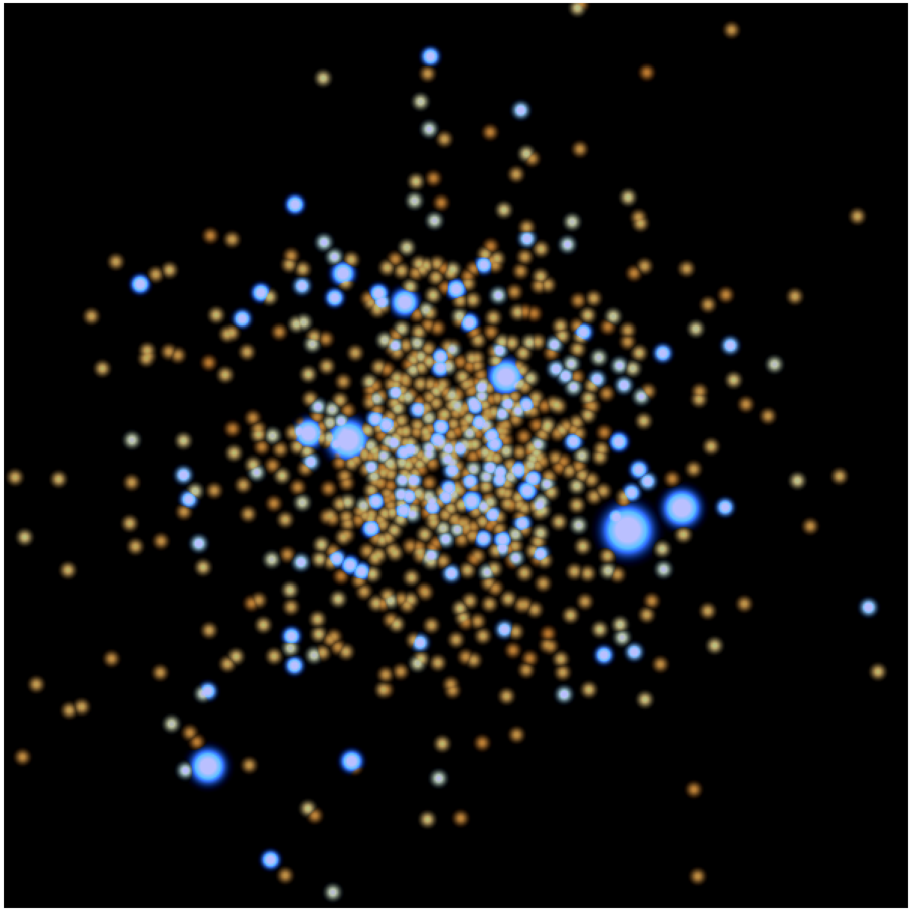
Total energy = 0 + Gaussian random motions;

Optional: seeded with turbulence as power spectrum.





TO GENERATE A STAR CLUSTER:



- Plummer or King profile for positions and velocities (already in starlab);

- Salpeter, Kroupa or similar IMF for masses;

- Fixed stellar radius or according to stellar evolution (SeBa, SSE);

- If stellar evolution, luminosities, temperatures and metallicity;

- Binaries: distribution for secondary masses, semi-major axis, eccentricities,...

TO GENERATE A GALAXY:

- Hernquist bulge as described;

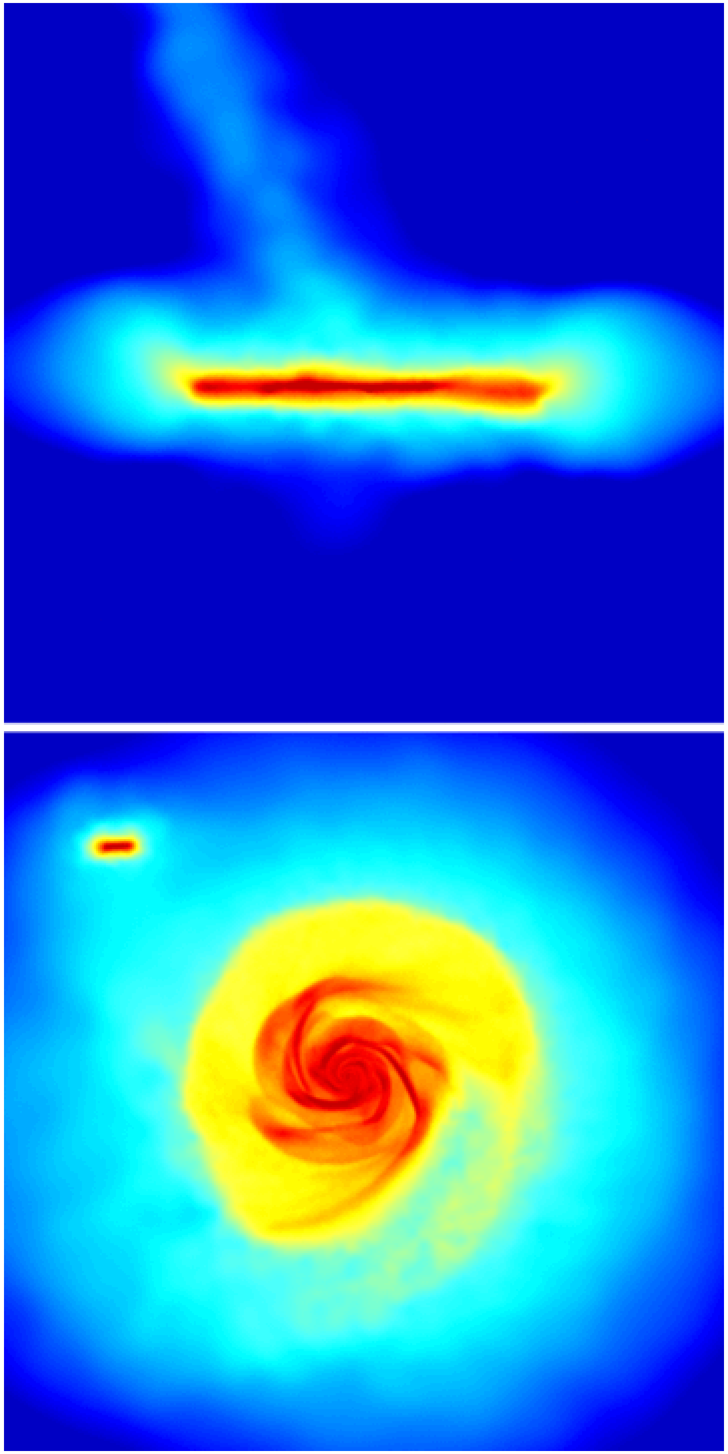
- Navarro, Frenk & White halo
  for dark matter;

- Eventually disc profile;

See e.g. Hernquist (1993),

Widrow & Dubinsky (2005),

ask me..



MM, Moore & Bland-Hawthorne 2008

COSMOLOGICAL SIMULATIONS:

Generate a cubic box with Zeldovich approximation

**SUGGESTION: use freely available grafic2 code,
developed by Edmund Bertschinger at MIT
see http://web.mit.edu/edbert**

Download grafic-2 tar file, read the README.txt (infos about settings), compile with Makefile

eg initial conditions are already in the right format for RAMSES