Michela Mapelli

N-body techniques for astrophysics:
Lecture 1 – General Introduction
Euler and Leaprofrog methods

PhD School in Astrophysics,
University of Padova
November 19 – 30 , 2018

A few words about the course:

The aim of this course:

to learn the basic concepts of state-of-the-art techniques of N-body simulations in astrophysics

At the end of the course you should be able to:

- **set up and run some very simple simulations;**
- **critically read and understand a paper about astrophysical N-body simulations**

A few words about the course:

Address:

Prof. Michela Mapelli,
University of Padova
email michela.mapelli@unipd.it
http://web.pd.astro.it/mapelli/

My team DEMOBLACK studies the formation channels of gravitational wave sources (black holes, neutron stars)

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1. Lectures will be published at
   http://web.pd.astro.it/mapelli/lectures.html
   and on Moodle
   https://elearning.unipd.it/dfa/enrol/index.php?id=439

   password: nb20Dot

2. Note: fill in the course evaluation sheet.
   It is anonymous, it helps me improving the lectures

3. ASK MANY QUESTIONS, it is important for you and me!!!!

A few words about the course:

<u>The EXAM:</u>

*** Each lecture will consist of an explanation section**
+ an exercise section

*** The final exam consists in the evaluation of the exercises done during the lectures**

*** You pass the exam if you do at least 60% of the exercises**

*** If you want help with the exercises, please write your codes in C, C++, python or fortran**

<u>The resources:</u>

you will have a guest account on our server

ssh -p1022 course@scighera.oapd.inaf.it -X

cd your_last_name/

BUT DON'T MESS UP WITH OUR SERVER, PLS

don't use the server for something not related to the course!

WHAT IS an N-Body SIMULATION?

?

WHAT IS an N-Body SIMULATION?

numerical integration of the forces acting on *N* particles for a time *t*

- ***astrophysics***
- ***fluid-dynamics***
- ***molecular dynamics***
- ***...***

WHAT IS an N-Body SIMULATION ** IN ASTROPHYSICS **?

numerical integration of the force of GRAVITY acting on *N* particles for a time *t*

WHAT IS an N-Body SIMULATION?

numerical integration of Newton equation

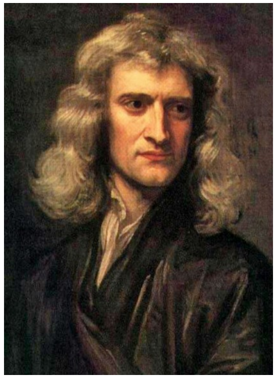
$$\ddot{\vec{r}}_i = -G \sum_{j \neq i} m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

Or the equivalent system of 2 x N x ndim 1st ord. differential eqs

$$\begin{cases} \dot{v}_i = -\sum \frac{G\, m_j}{r_{ij}^3} x_{ij} \\ \dot{x}_i = v_i \end{cases}$$

<u>DOES IT HAVE ANALYTIC SOLUTION?</u>

WHAT IS an N-Body SIMULATION?



Newton

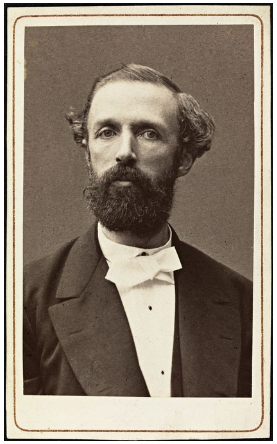
1687: Newton finds the equation

$$\ddot{\vec{r}}_i = -G \sum_{j \neq i} m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

1710: Bernoulli derives analytic solution for N = 2

Bernoulli

1885: a challenge was proposed (to be answered before 21/01/1889) in honor of the 60th birthday of King Oscar II of Sweden and Norway.



Oscar II of Sweden

'Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.'

NOBODY FOUND THE SOLUTION

1991: Qiudong Wang finds a convergent power series solution for a generic number of bodies. Mathematically correct, but too difficult and slow convergence

Q. Wang

http://www.mittag-leffler.se/library/henri-poincare

WHAT IS an N-Body SIMULATION?

- Analytic problem only for N = 2 (and restricted N = 3)

- gravity force does not fade off (even far away particle interact)
  → calculation of a system of N particles cannot be decomposed in smaller pieces

$$\begin{cases} \dot{v}_i = -\sum \frac{G\, m_j}{r_{ij}^3} x_{ij} \\ \dot{x}_i = v_i \end{cases}$$

A system of 2 x *N* x Ndim differential equations
with NO known analytic solution

→ FIND A NUMERICAL SOLUTION

WHAT WOULD YOU DO? ?

WHAT IS an N-Body SIMULATION?

TAYLOR EXPANSION

$$x_i(t+\Delta t) = x_i(t) + \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\,\Delta t + \frac{1}{2}\,\frac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2}\,\Delta t^2 + O(\Delta t^3)$$

$$v_i(t+\Delta t) = v_i(t) + \frac{\mathrm{d}v_i(t)}{\mathrm{d}t}\,\Delta t + \frac{1}{2}\,\frac{\mathrm{d}^2 v_i(t)}{\mathrm{d}t^2}\,\Delta t^2 + O(\Delta t^3)$$

Predicting time t +Δt with info at time t

Truncation order gives error order

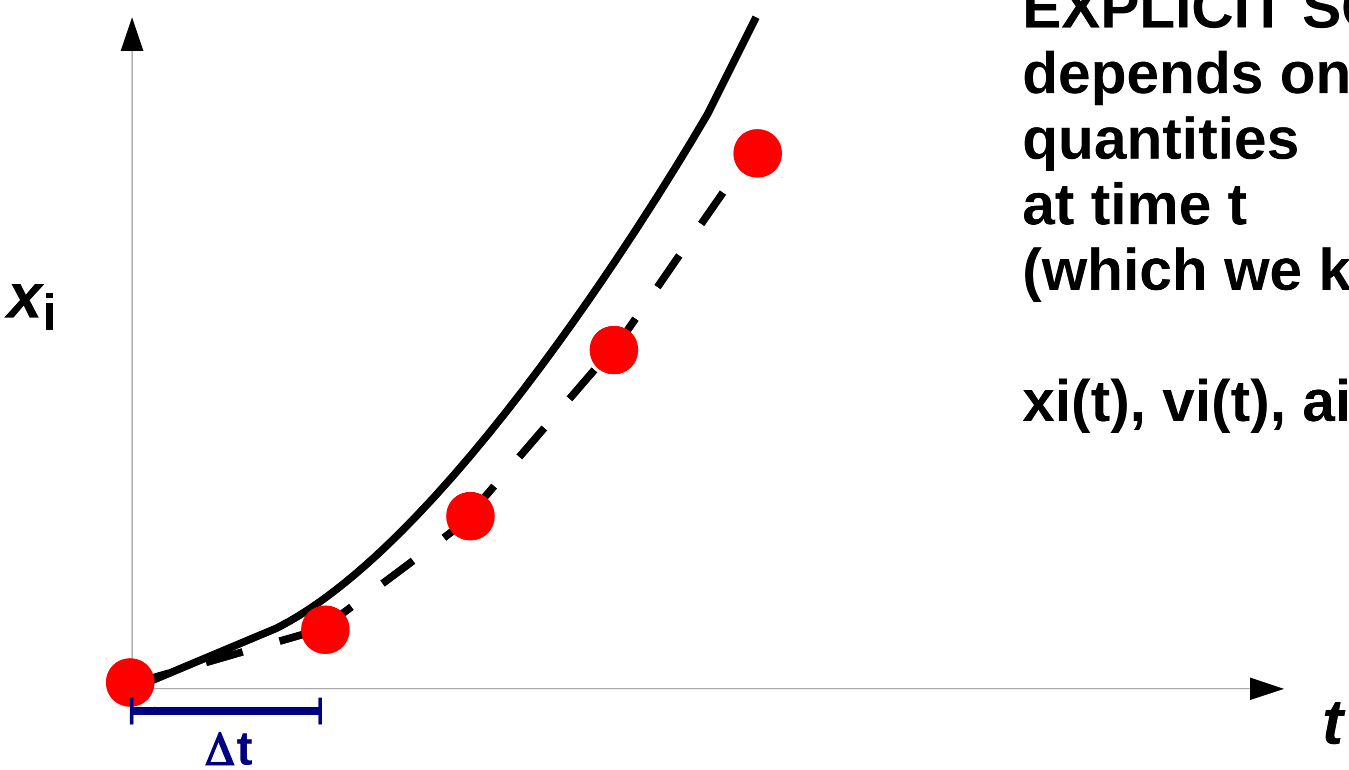
A 1st order method has Δt order errors
A 2nd order method has Δt^2 order errors
A 3rd order method has Δt^3 order errors
...

EULER METHOD:

Taylor expansion at 1st order

$$x_i(t+\Delta t) = x_i(t) + \frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\Delta t$$

$$v_i(t+\Delta t) = v_i(t) + \frac{\mathrm{d}v_i(t)}{\mathrm{d}t}\Delta t$$



EXPLICIT SCHEME = depends only on quantities at time t (which we know)

xi(t), vi(t), ai(t)

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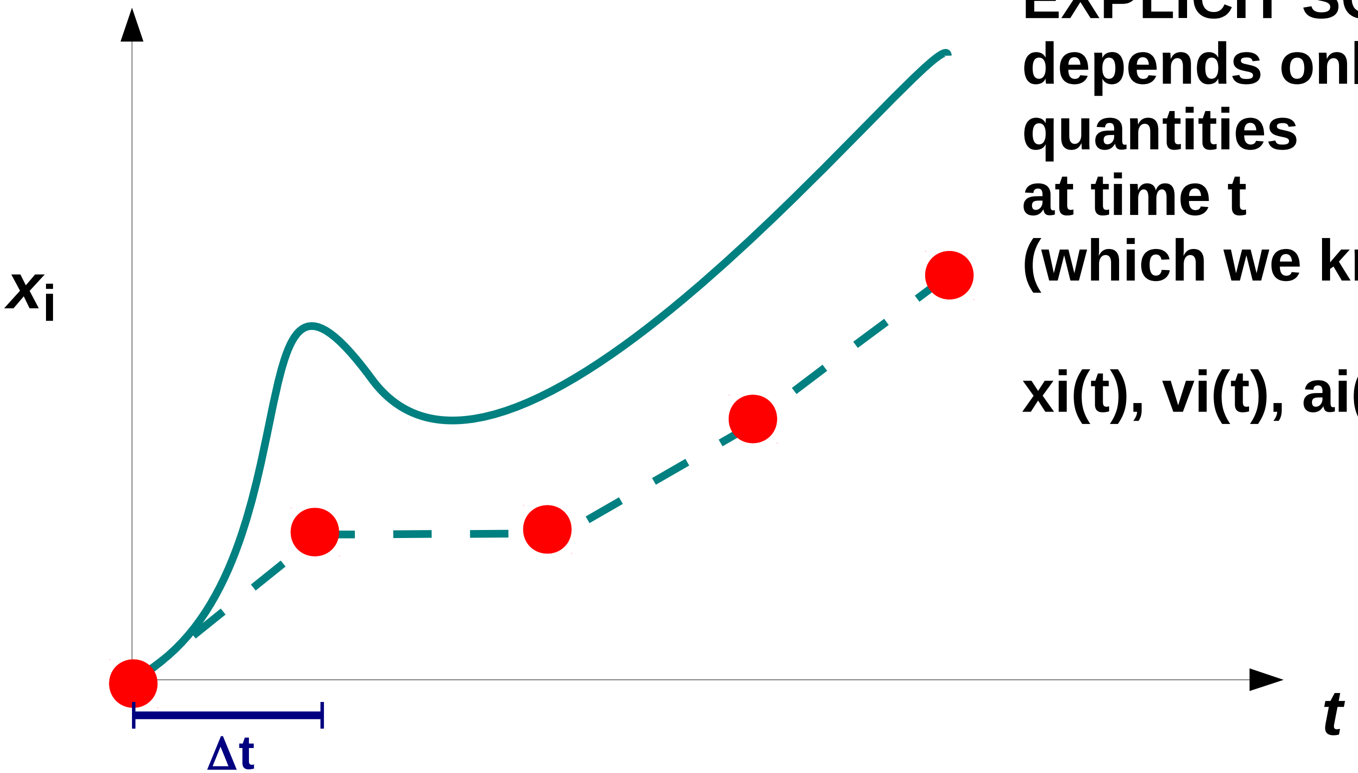


EXPLICIT SCHEME = depends only on quantities at time t (which we know)

xi(t), vi(t), ai(t)

EULER METHOD:

Exercise # 1:

calculate motion of two bodies in a binary system with Euler method

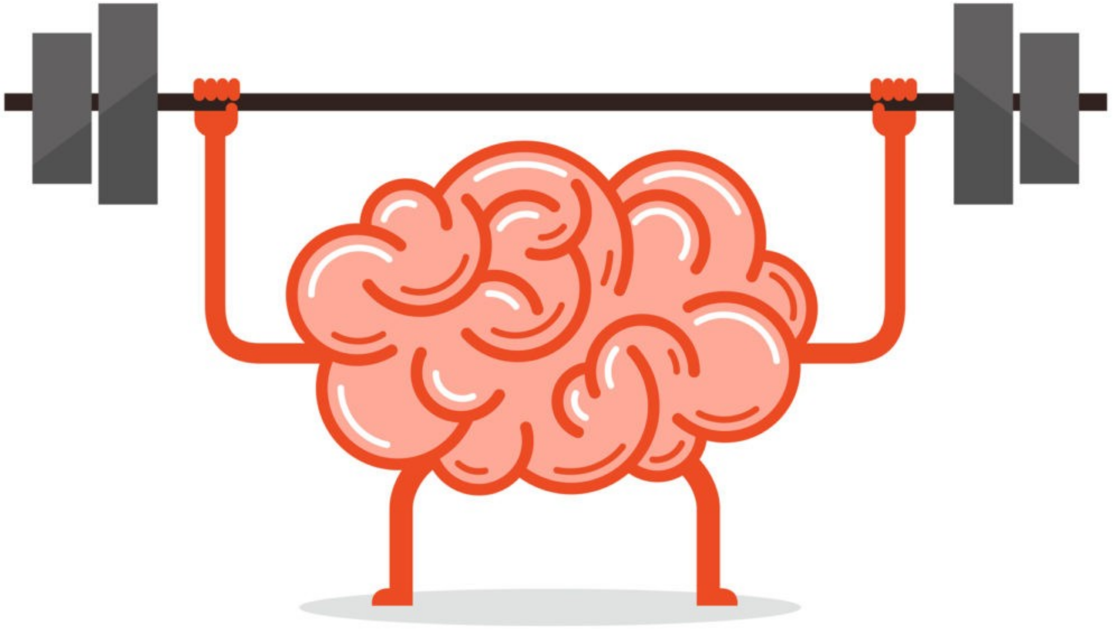
Initial conditions:

X1 = 1.0
Y1 = 1.0

Vx1 = -0.5
Vy1 = 0.0

X2 = -1.0
Y2 = -1.0

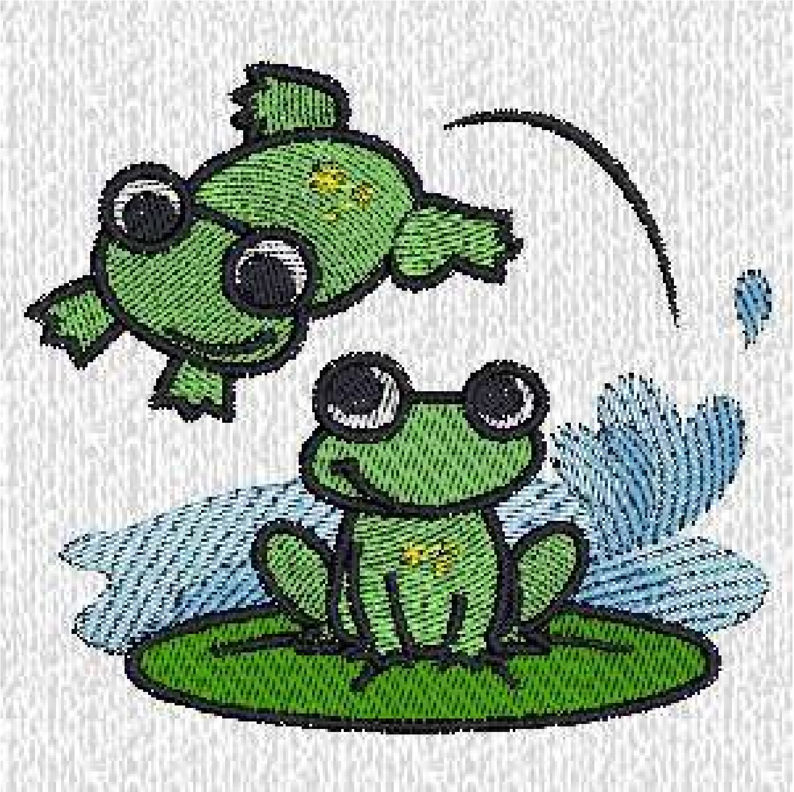
Vx2 = 0.5
Vy2 = 0.0



Assume m1 = m2 = 1, G = 1

LEAPFROG METHOD:

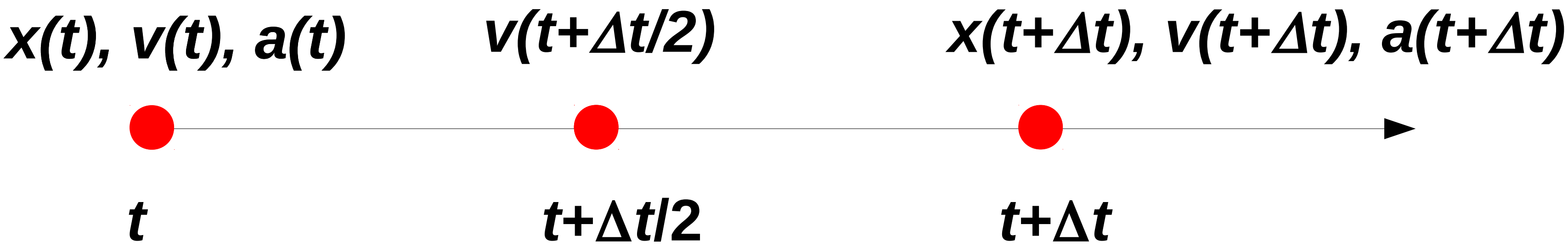
The name comes from the leapfrog game



IDEA: evaluation of velocity and position jumps like little happy frogs within a timestep..

LEAPFROG METHOD:

same as Euler but evaluated in between a timestep



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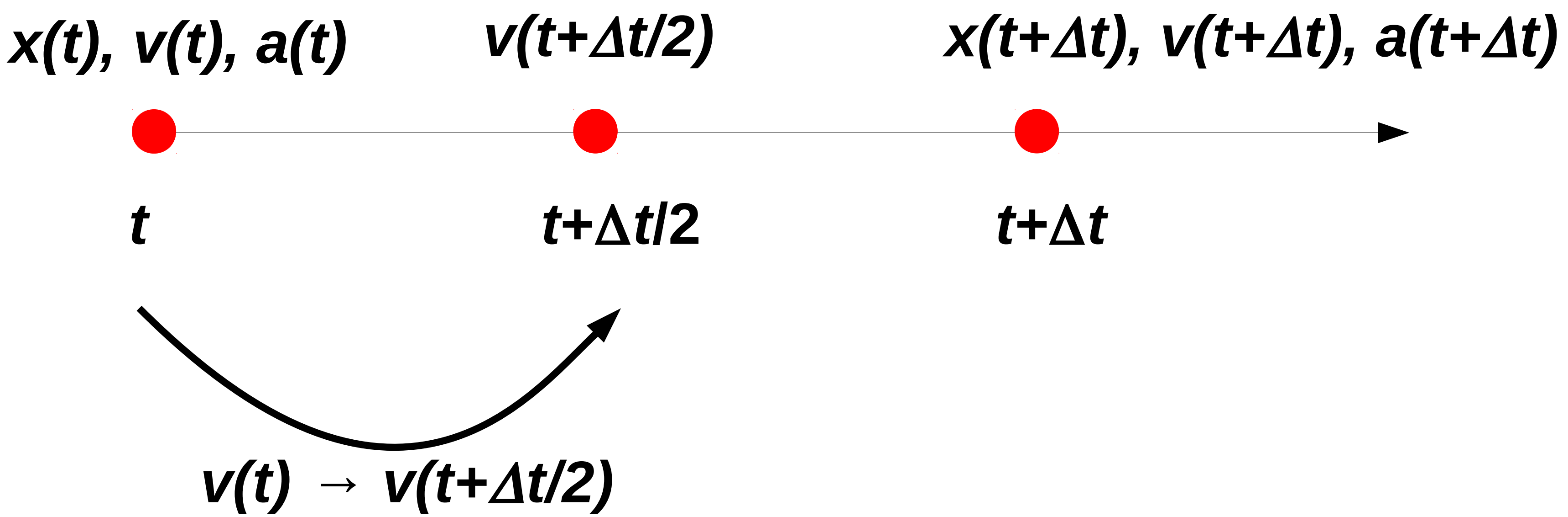

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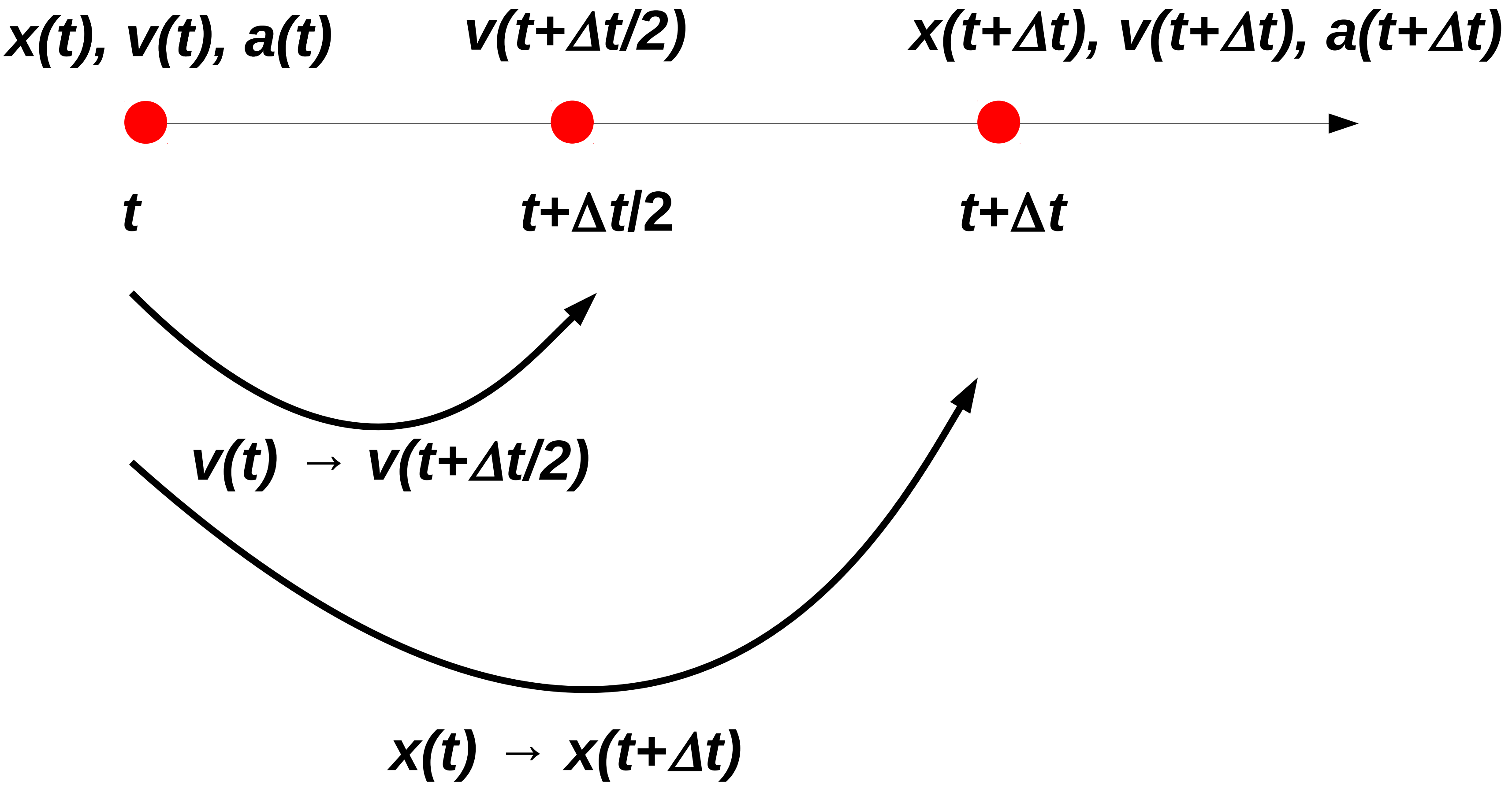

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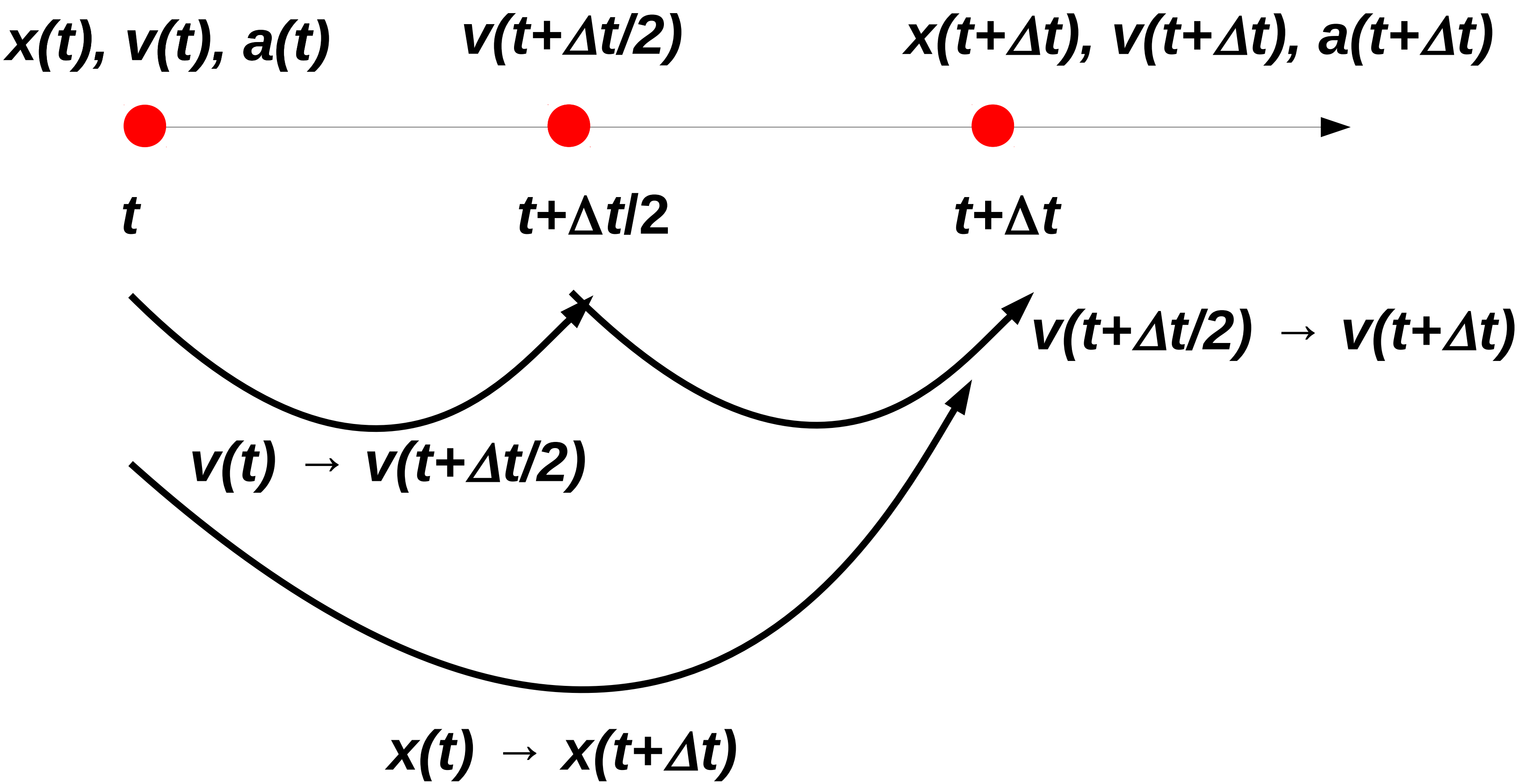

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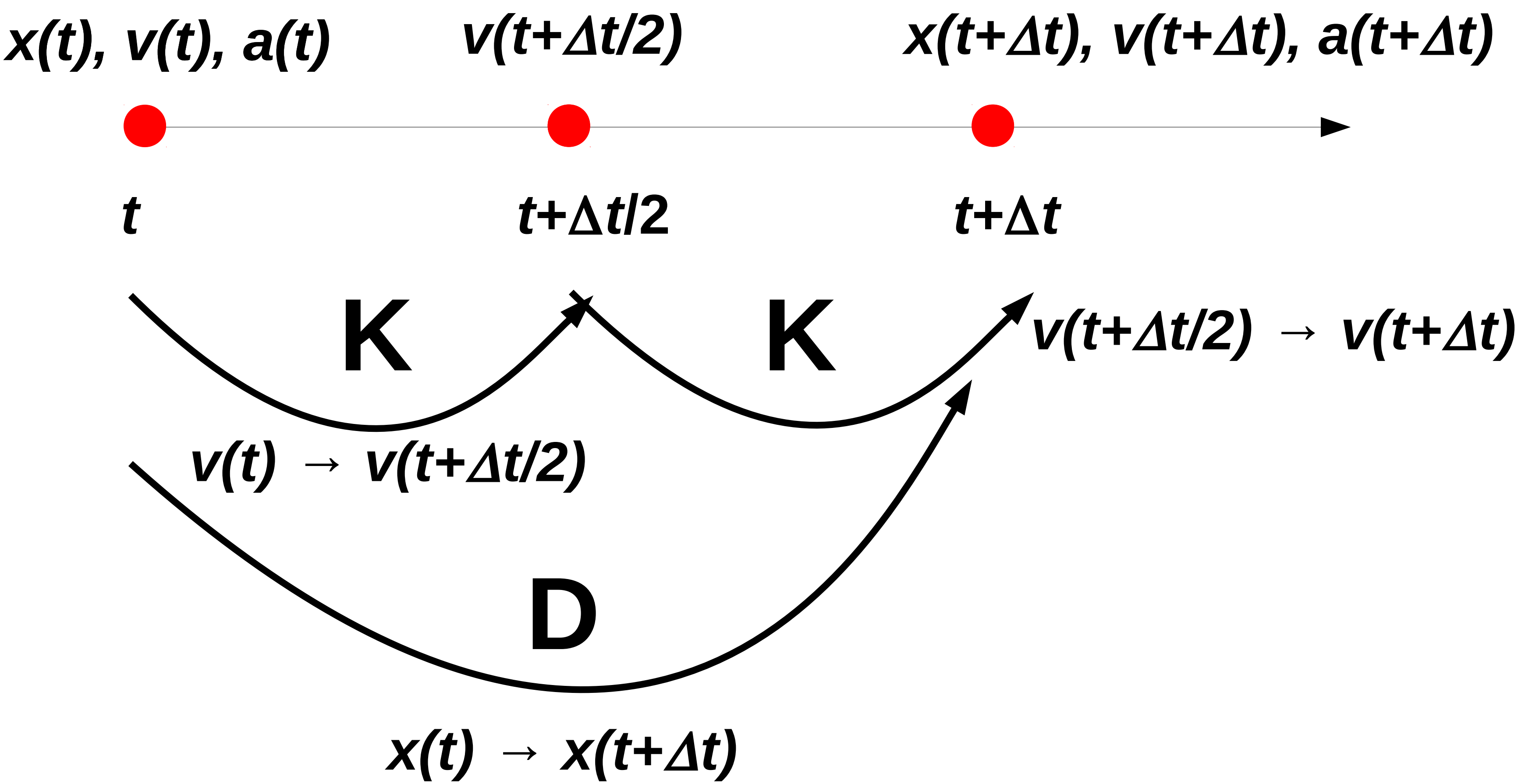
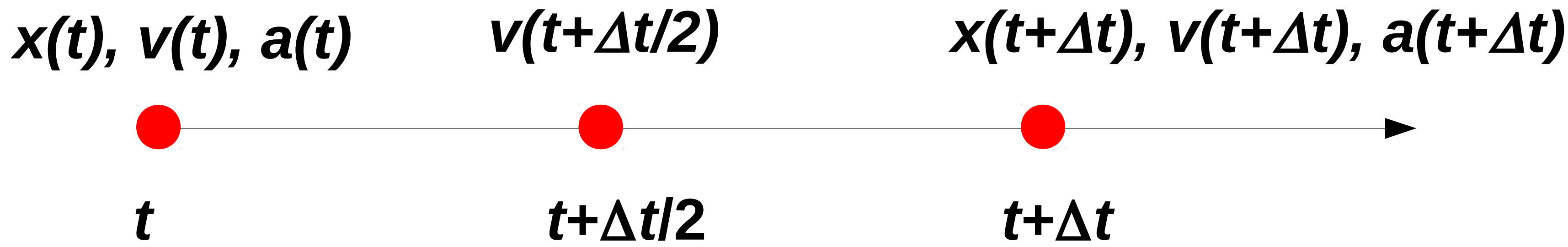


Kick + Drift + Kick (KDK) scheme

LEAPFROG METHOD:

same as Euler but evaluated in between a timestep



$$v\left(t+\frac{\Delta t}{2}\right) = v(t) + \frac{\Delta t}{2}\,a(t)$$

$$x(t+\Delta t) = x(t) + v\left(t+\frac{\Delta t}{2}\right)\Delta t$$

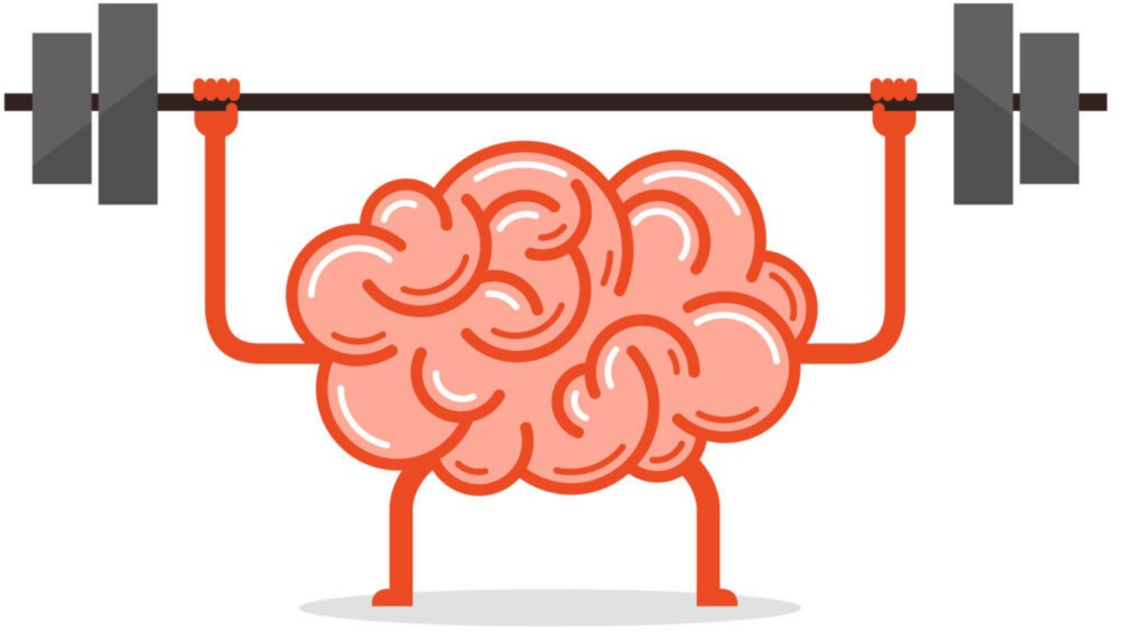
$$v(t+\Delta t) = v\left(t+\frac{\Delta t}{2}\right) + \frac{1}{2}\Delta t\,\underline{a(t+\Delta t)}$$

$$\longrightarrow \quad x(t+\Delta t) = x(t) + v(t)\,\Delta t + \frac{1}{2}a(t)\Delta t^2$$

$$v(t+\Delta t) = v(t) + \frac{1}{2}a(t)\,\Delta t + \frac{1}{2}a(t+\Delta t)\,\Delta t$$

LEAPFROG METHOD:

Exercise # 2:
calculate motion of two bodies in a binary system with Leapfrog method



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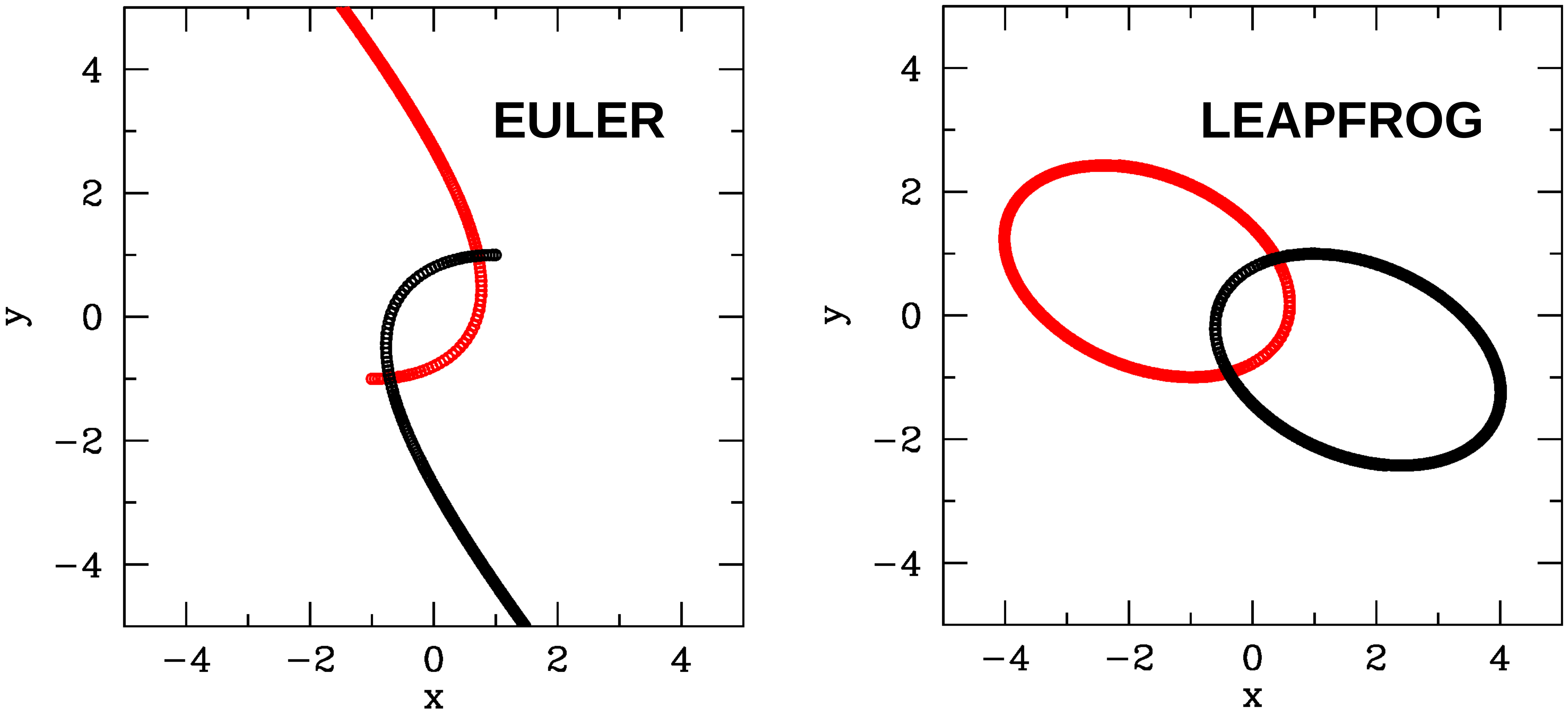
Vx2 = 0.5
Vy2 = 0.0

Assume m1 = m2 = 1, G = 1

and calculate energy conservation in Leapfrog and Euler case

EULER vs LEAPFROG METHOD: a simple test

Both Euler and Leapfrog are 2d order methods, but..



Same initial conditions: integration of a Keplerian binary

EULER vs LEAPFROG METHOD: a simple test

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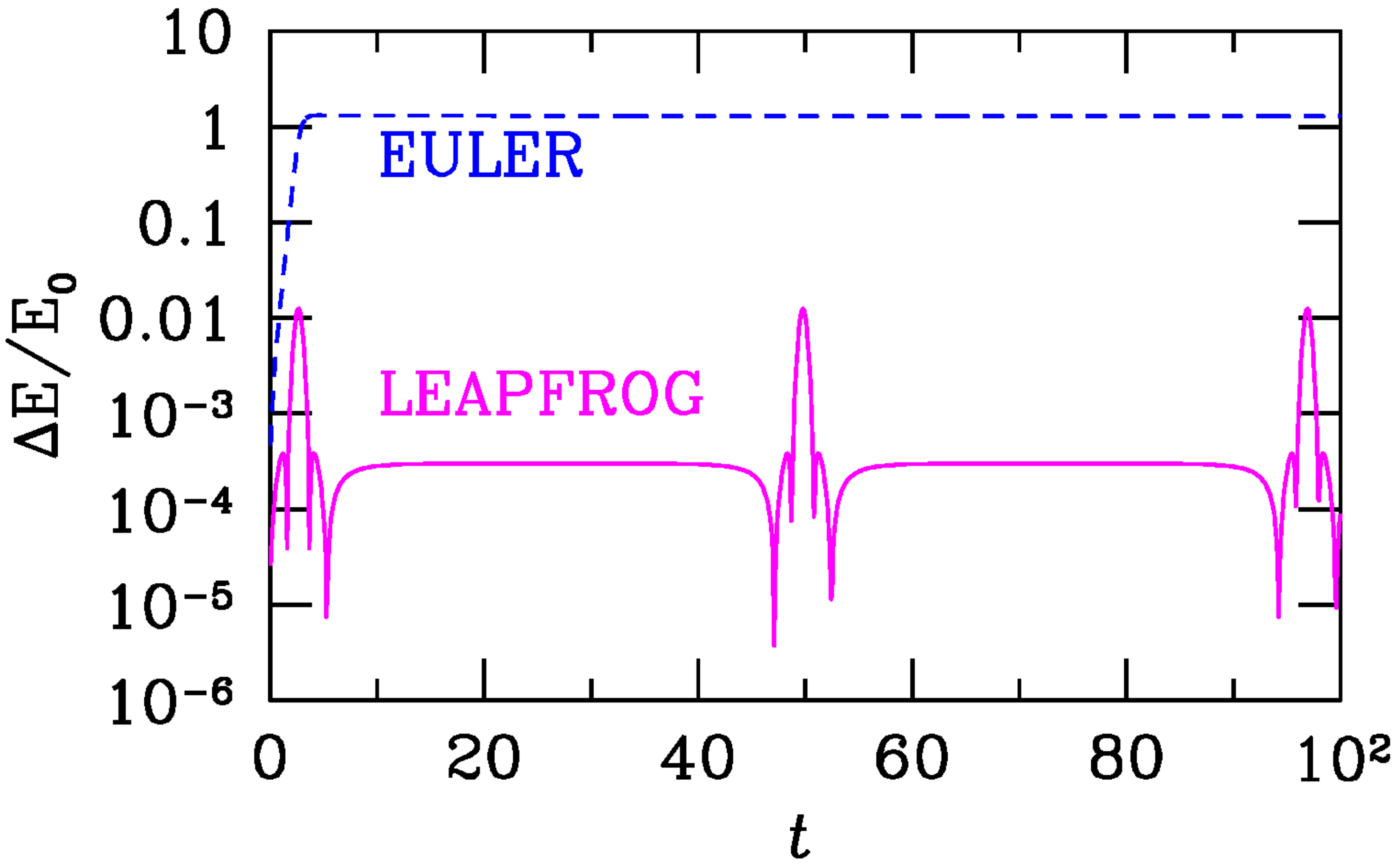


Same initial conditions: integration of a Keplerian binary

N-body UNITS:

MOST N-body codes work in N-body units

since Gravity is the main force, choose units so that G=1

DEFINITION OF CODE UNITS (SO THAT $G=1$)

$$6.67 \times 10^{-8} \frac{cm^3}{s^2 g} \simeq G = \frac{U_l^3}{U_t^2 \, U_m}$$

$$\Rightarrow U_t^2 = \frac{U_l^3}{G \, U_m}$$

N-body UNITS:

DEFINITION of MSCALE, TSCALE, LSCALE

$$M_{scale} := \frac{1}{U_m}$$

$$T_{scale} := \frac{1}{U_t}$$

$$L_{scale} := \frac{1}{U_L}$$

N-body UNITS:

FROM CODE UNITS TO PHYSICAL UNITS and VICEVERSA:

$$M_{code} = M_{phys} \, M_{scale} = \frac{M_{phys}}{M_{TOT}}$$

- M_{code}: mass of particles in the code (mass in code units)
- M_{phys}: mass of particles in physical units
- $M_{scale} := \frac{1}{m_{TOT}}$

$$T_{code} = T_{phys} \, T_{scale} = \frac{T_{phys}}{0.25 \, Myr}$$

- T_{code}: time in code units
- $T_{scale} := \frac{1}{U_t} = \frac{1}{0.25 \, Myr}$ (in our example)

$$L_{code} = L_{phys} \, L_{scale} = \frac{L_{phys}}{1 \, pc}$$

- L_{code}: length in code units
- $L_{scale} := \frac{1}{U_L} = \frac{1}{1 \, pc}$ (in our example: $\frac{1}{1 \, pc}$)

$$V_{code} = V_{phys} \, \frac{L_{scale}}{T_{scale}} = V_{phys} \, \frac{U_T}{U_L} = V_{phys} \, \frac{0.25 \, Myr}{1 \, pc}$$

- V_{code}: velocity in code units

N-body UNITS:

What is the advantage of N-body units?

N-body UNITS:

What is the advantage of N-body units?

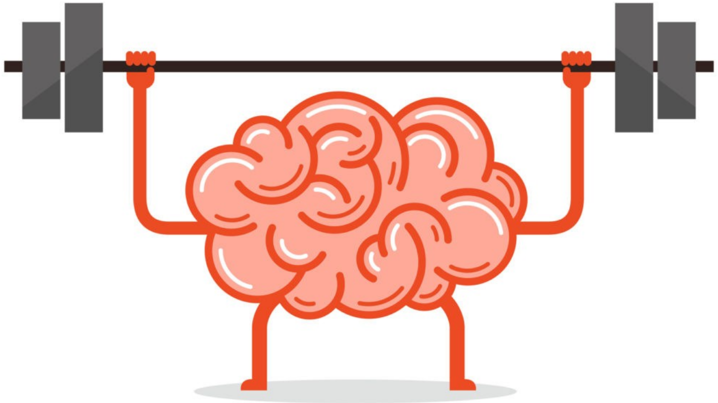
1. **choosing right units helps the integrator!**
   Rounding errors' importance is different
   if input quantities are 10^-30, 10^45
   rather than 1, 2

2. **your N-body problems is perfectly SCALABLE**
   (unless stellar evolution or gas are involved,
   i.e. dissipative processes)

N-body UNITS:

Exercise # 3:

Rewrite codes of exercises #1 and #2 in N-body units for the following system



M1 = 1.989e33 g

M2 = 5.972e27 g

semi-major axis of the minor body = 1.496e13 cm

eccentricity = 0

Remember that G = 6.667e-8 $cm^3\ g^{-1}\ s^{-2}$

N-body UNITS:

Exercise # 3b:

Write a Leapfrog scheme for N particles with N>~10 and arbitrary initial conditions