

# **Dynamics of Stars and Black Holes in Dense Stellar Systems:**

## **Lecture III:**

### **BINARIES and 3-BODY ENCOUNTERS**

# BINARIES as ENERGY RESERVOIR:

Binaries have a energy reservoir (their internal energy) that can be exchanged with stars.

INTERNAL ENERGY: total energy of the binary – kinetic energy of the centre-of-mass

$$E_{int} = \frac{1}{2} \mu v^2 - \frac{G m_1 m_2}{r}$$

where  $m_1$  and  $m_2$  are the mass of the primary and secondary member of the binary,  $\mu$  is the reduced mass ( $:= m_1 m_2 / (m_1 + m_2)$ ).  
 $r$  and  $v$  are the relative separation and velocity.

$E_{int} < 0$  if the binary is bound

Note that  $E_{int}$  can be interpreted as the **energy of the 'reduced particle'**:  
a fictitious particle of mass  $\mu$  orbiting in the potential  $- G m_1 m_2 / r$

# BINARIES as ENERGY RESERVOIR:

As far as the binary is bound, the orbit of the reduced particle is a Kepler ellipse with semi-major axis  $a$ . Thus, the energy integral of motion is

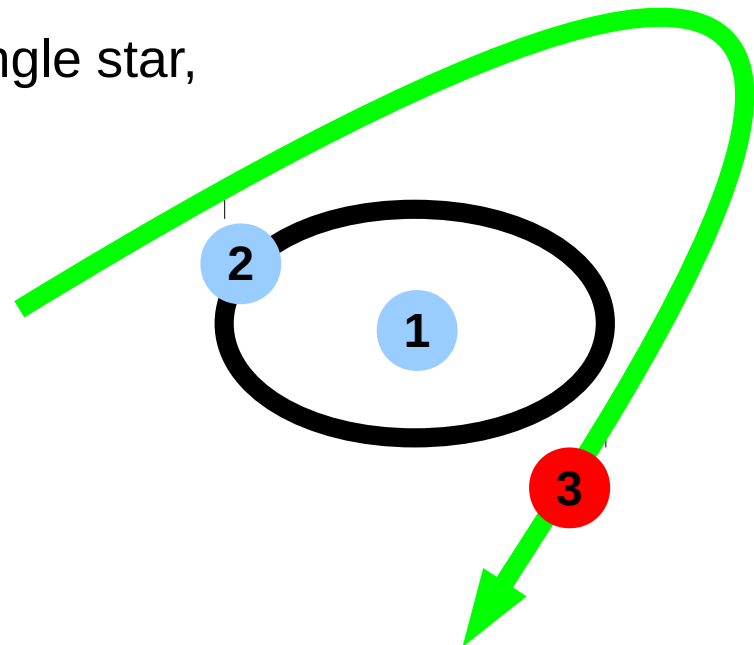
$$E_{int} = -\frac{G m_1 m_2}{2 a} = -E_b$$

where  $E_b$  is the **BINDING ENERGY** of the binary.

## **THE ENERGY RESERVOIR of BINARIES can be EXCHANGED with stars:**

during a **3-BODY INTERACTION**,  
i.e. an interaction between a binary and a single star,  
the single star can either

- EXTRACT INTERNAL ENERGY from the binary,
- or lose a fraction of its kinetic energy, which is converted into internal energy of the binary.



# BINARIES as ENERGY RESERVOIR:

If the star extracts  $E_{int}$  from the binary, its final kinetic energy ( $K_f$ ) is higher than the initial kinetic energy ( $K_i$ ).

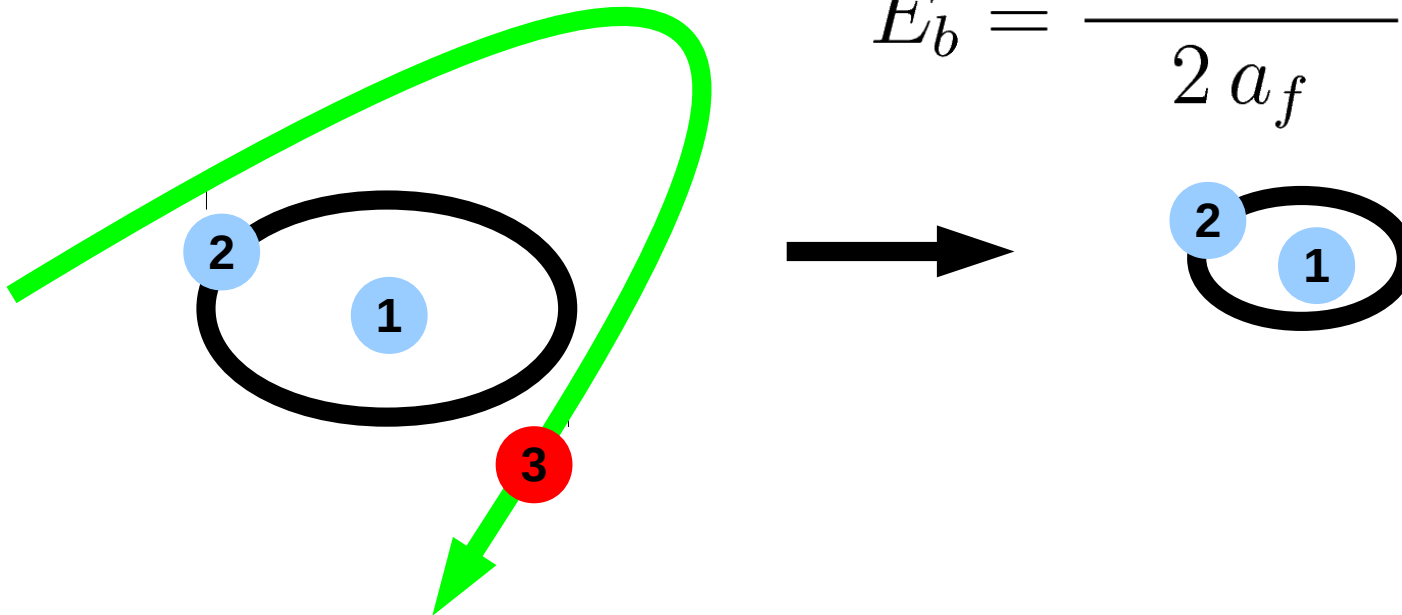
To better say:  $K_f$  of the centres-of-mass of the single star and of the binary is higher than their  $K_i$ .

We say that the STAR and the BINARY acquire **RECOIL VELOCITY**.

$E_{int}$  becomes more negative, i.e.  $E_b$  higher: the binary becomes more bound (e.g.  $a$  decreases or  $m_1$  and  $m_2$  change).

$$a_f < a_i$$

$$E_b = \frac{G m_1 m_2}{2 a_f} > \frac{G m_1 m_2}{2 a_i}$$



CARTOON of a FLYBY ENCOUNTER where  $a_f < a_i \rightarrow E_b$  increases

# BINARIES as ENERGY RESERVOIR:

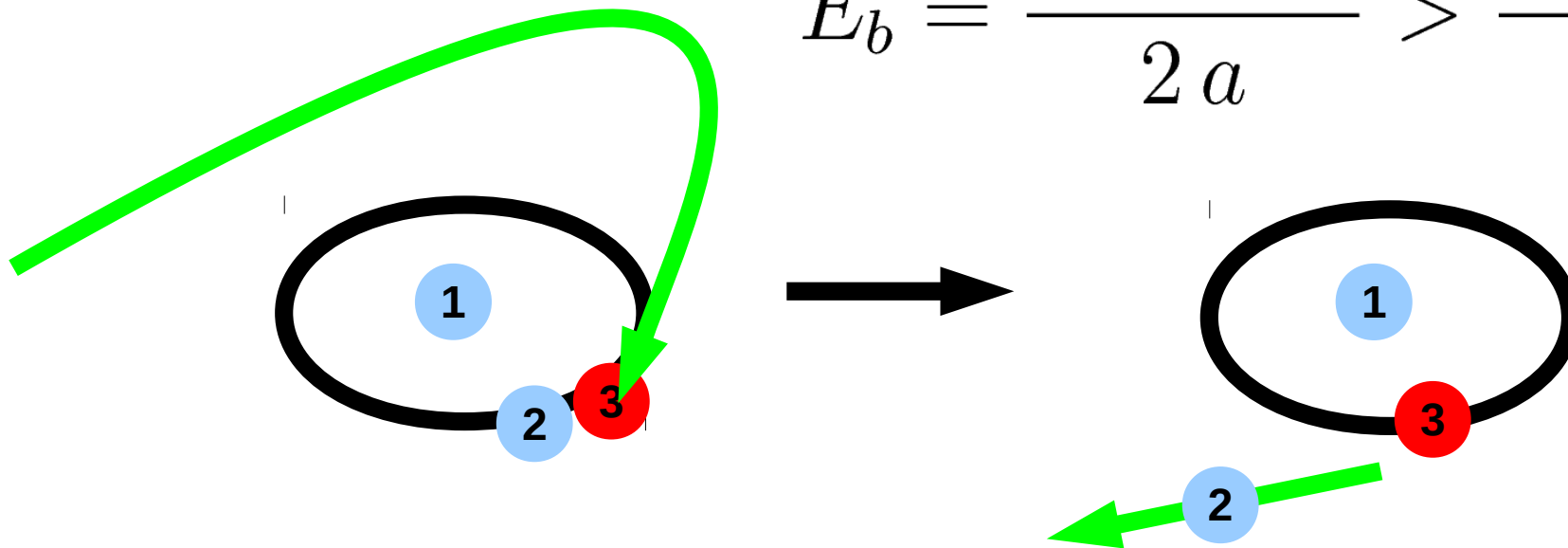
An alternative way for a binary to transfer internal energy to field stars and increase its binding energy  $E_b$  is an **EXCHANGE**:

the single star replaces one of the former members of the binary.

An exchange interaction is favored when the mass of the single star  $m_3$  is HIGHER than the mass of one of the members of the binary so that the new  $E_b$  of the binary is higher than the former:

$$m_3 > m_2$$

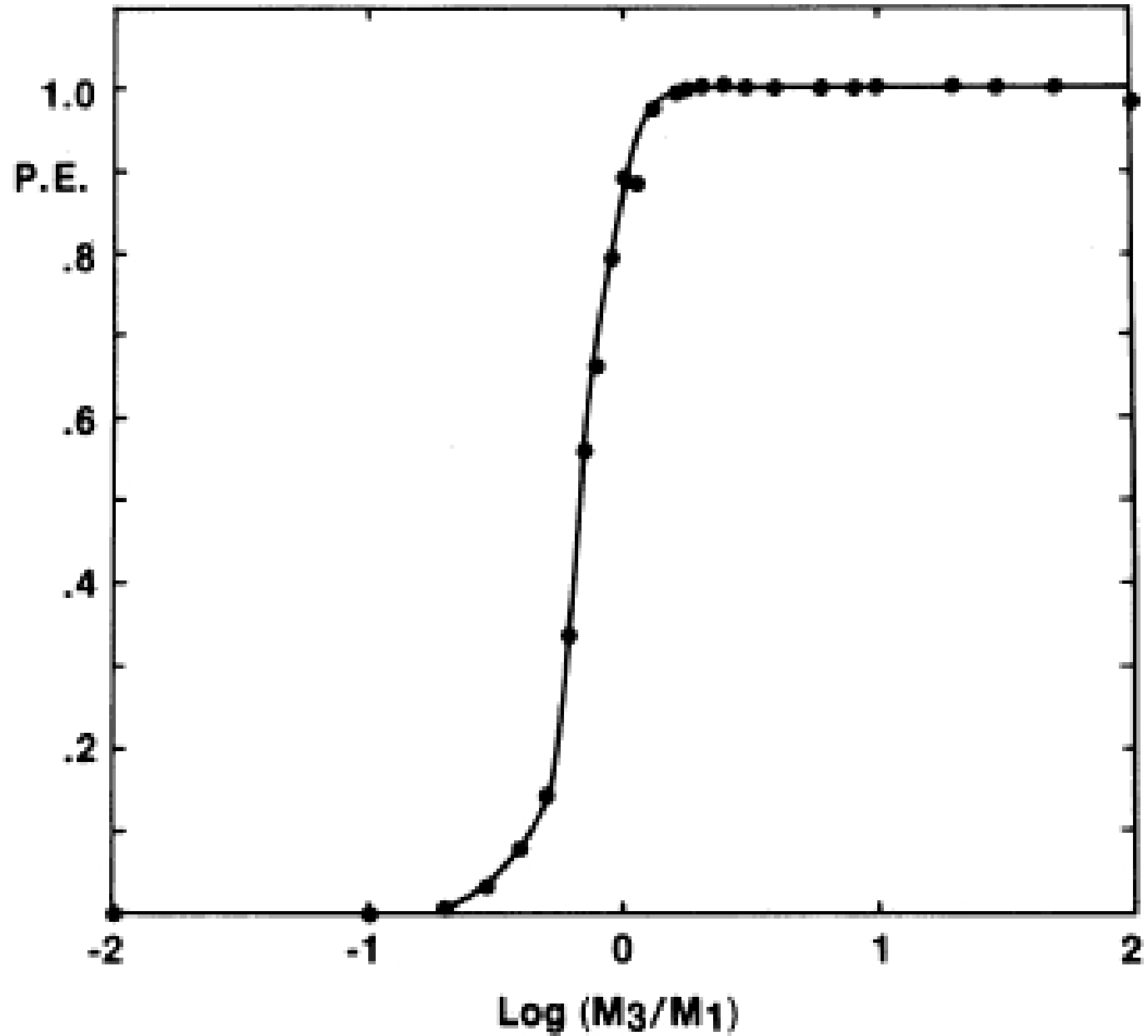
$$E_b = \frac{G m_1 m_3}{2 a} > \frac{G m_1 m_2}{2 a}$$



CARTOON of a EXCHANGE ENCOUNTER where  $m_3 > m_2 \rightarrow E_b$  increases

# EXCHANGE PROBABILITY:

Probability  
increases  
dramatically  
if  $m_3 \geq m_1$

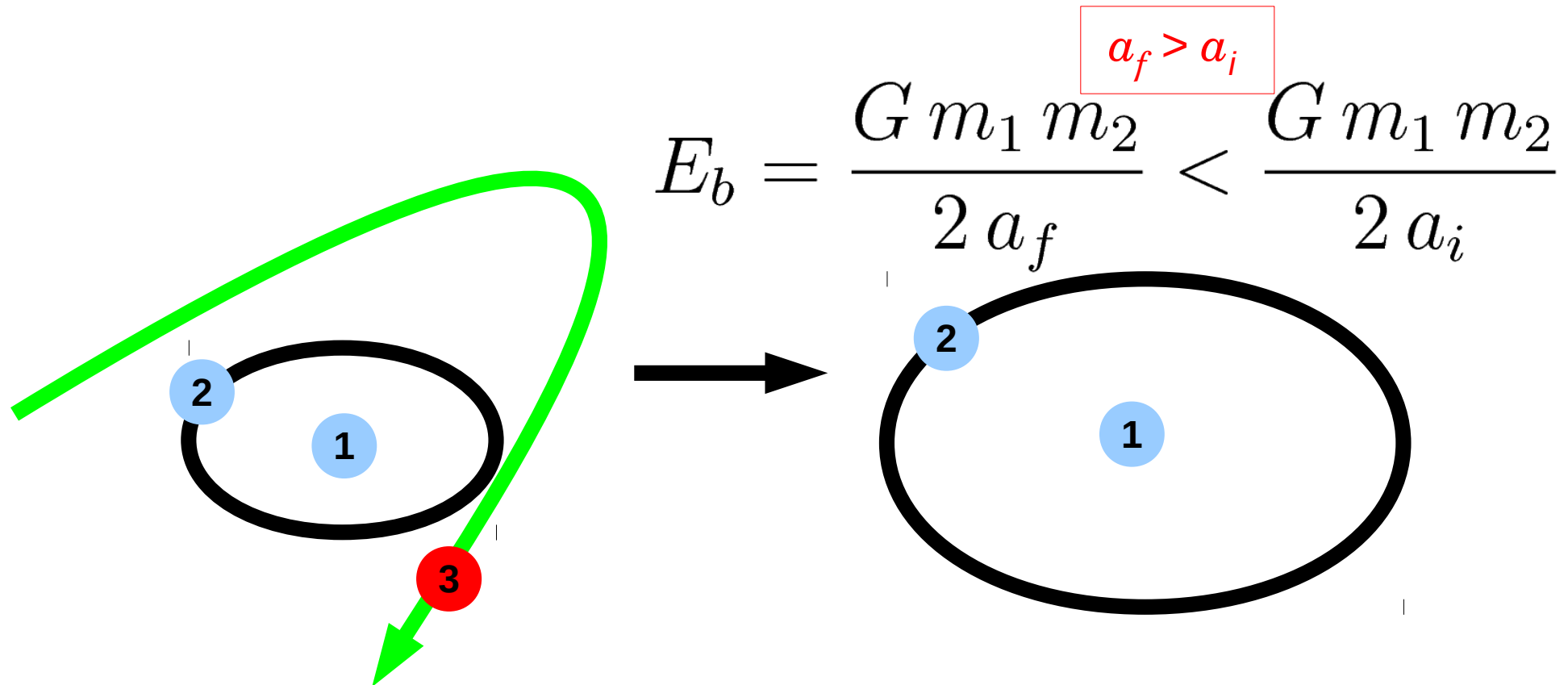


Hills & Fullerton 1980, AJ, 85, 1281

# BINARIES as ENERGY RESERVOIR:

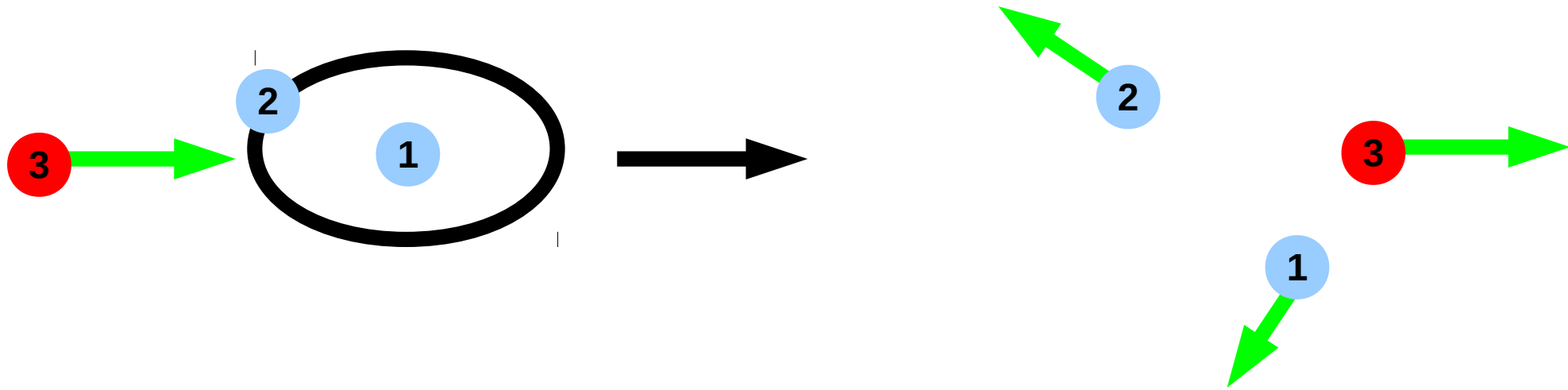
If the star transfers kinetic energy to the binary, its final kinetic energy ( $K_f$ ) is obviously lower than the initial kinetic energy ( $K_i$ ). To better say:  $K_f$  of the centres-of-mass of the single star and of the binary is lower than their  $K_i$ .

$E_{int}$  becomes less negative, i.e.  $E_b$  smaller: the binary becomes less bound (e.g.  $a$  increases) or is even **IONIZED** (:= becomes **UNBOUND**).



CARTOON of a FLYBY ENCOUNTER where  $a_f > a_i \rightarrow E_b$  decreases

# BINARIES as ENERGY RESERVOIR:



A single star can IONIZE the binary only if its velocity at infinity (=when it is far from the binary, thus unperturbed by the binary) exceeds the CRITICAL VELOCITY (Hut & Bahcall 1983, ApJ, 268, 319)

$$v_c = \sqrt{\frac{G m_1 m_2 (m_1 + m_2 + m_3)}{m_3 (m_1 + m_2) a}}$$

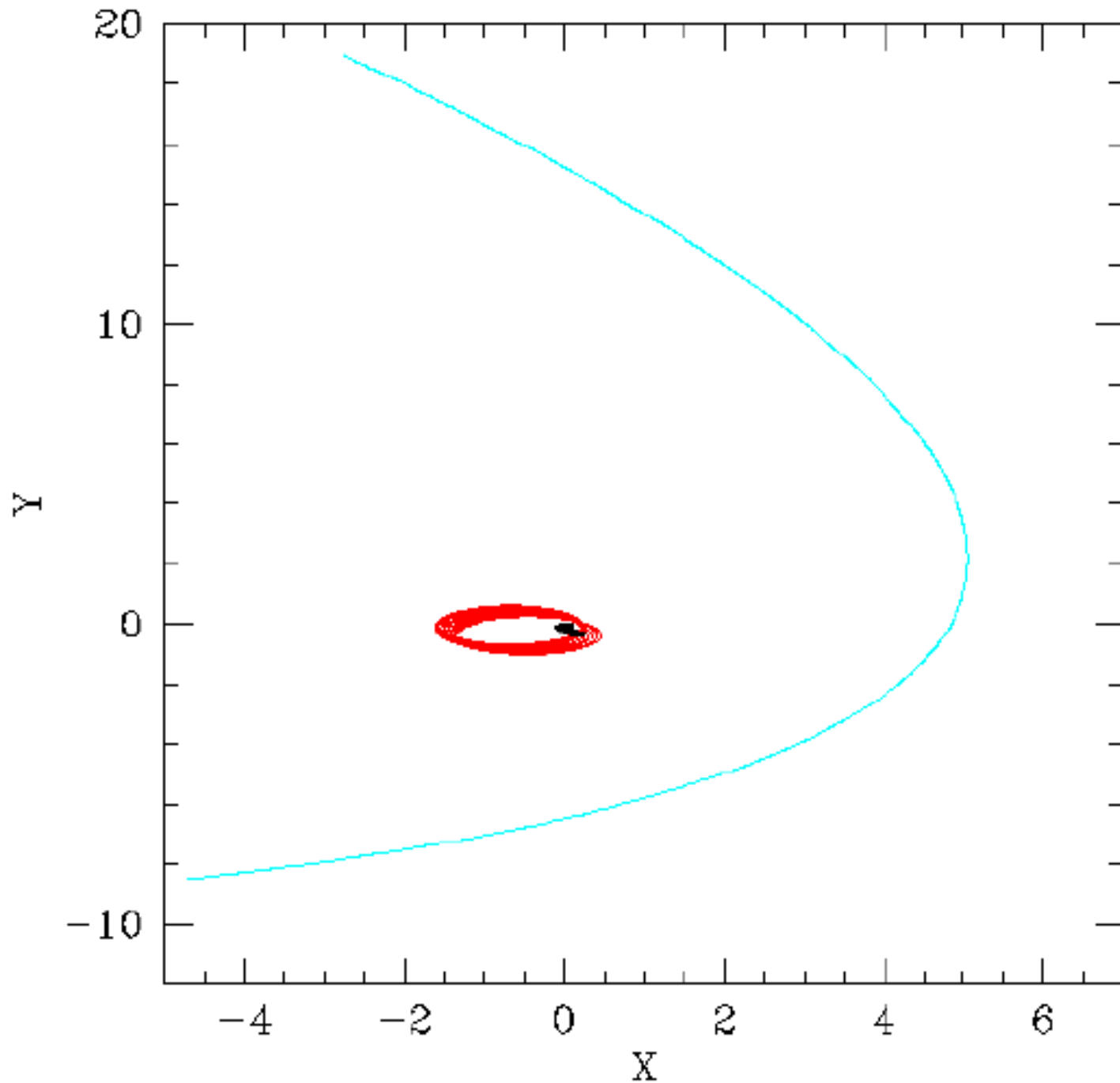
This critical velocity was derived by imposing that the K of the reduced particle of the 3-body system is equal to  $E_b$ :

$$\frac{1}{2} \frac{m_3 (m_1 + m_2)}{(m_1 + m_2 + m_3)} v_c^2 = \frac{G m_1 m_2}{2 a}$$



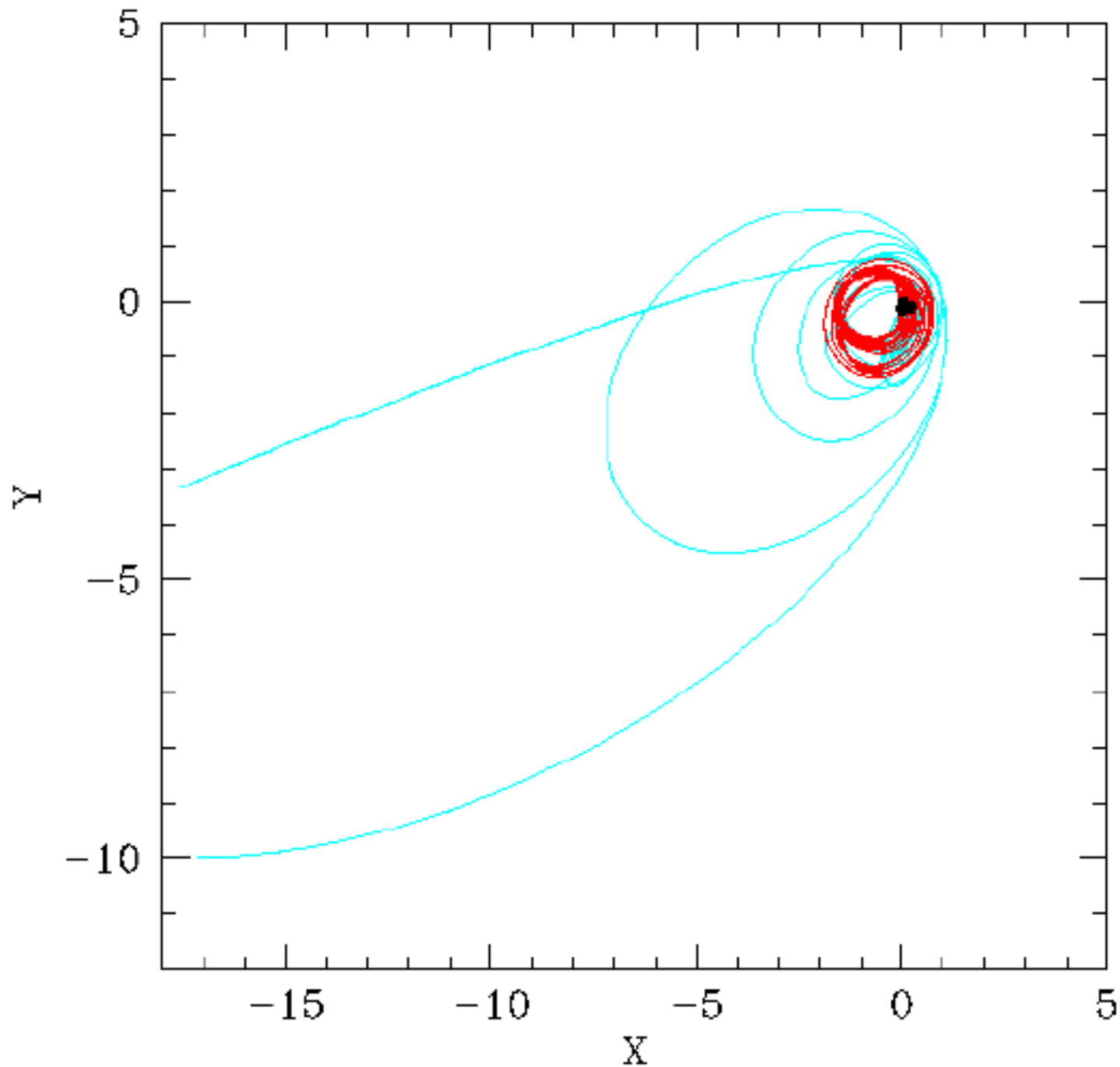
# EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

PROMPT  
FLYBY:



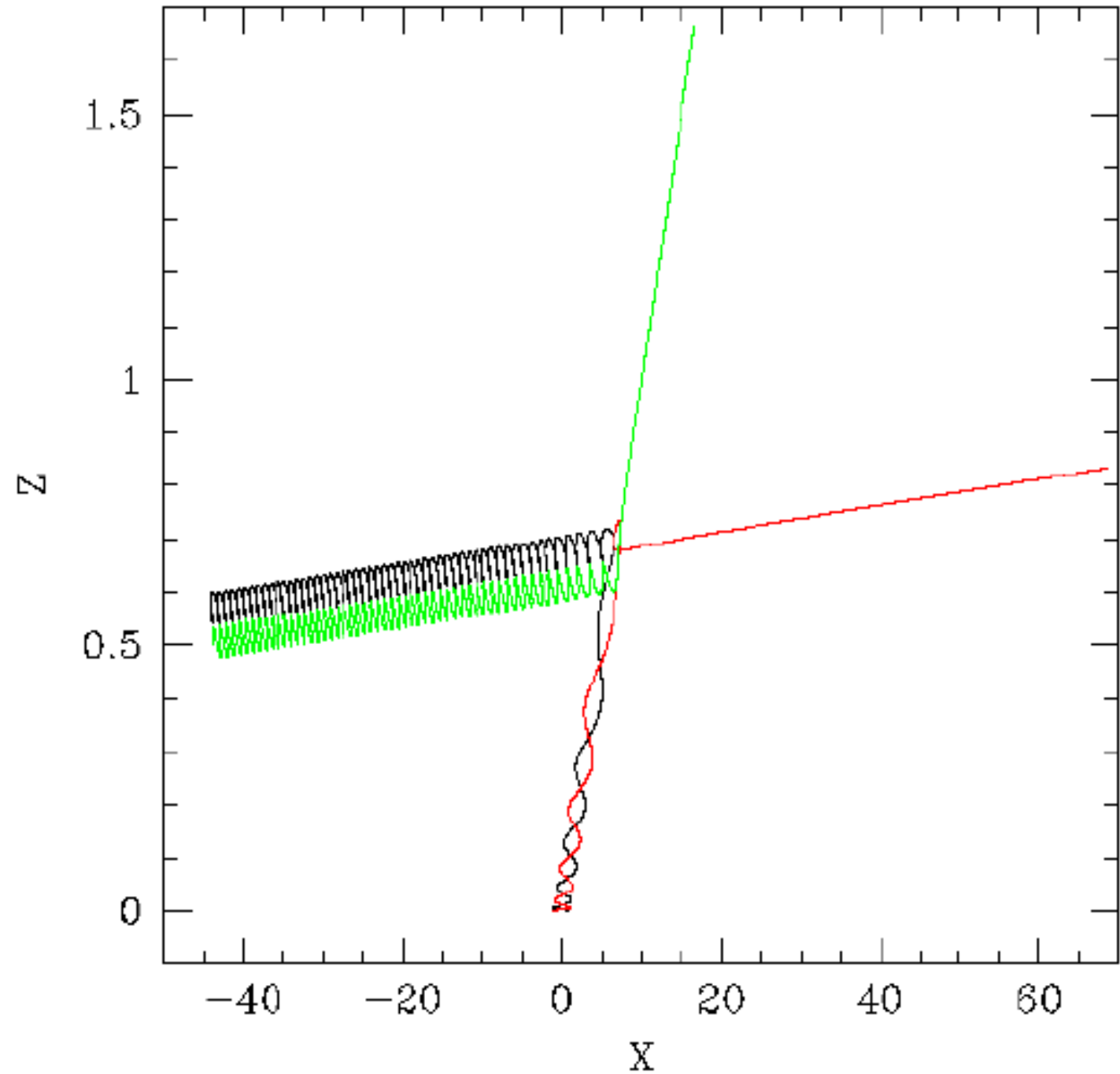
# EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

RESONANT  
FLYBY:



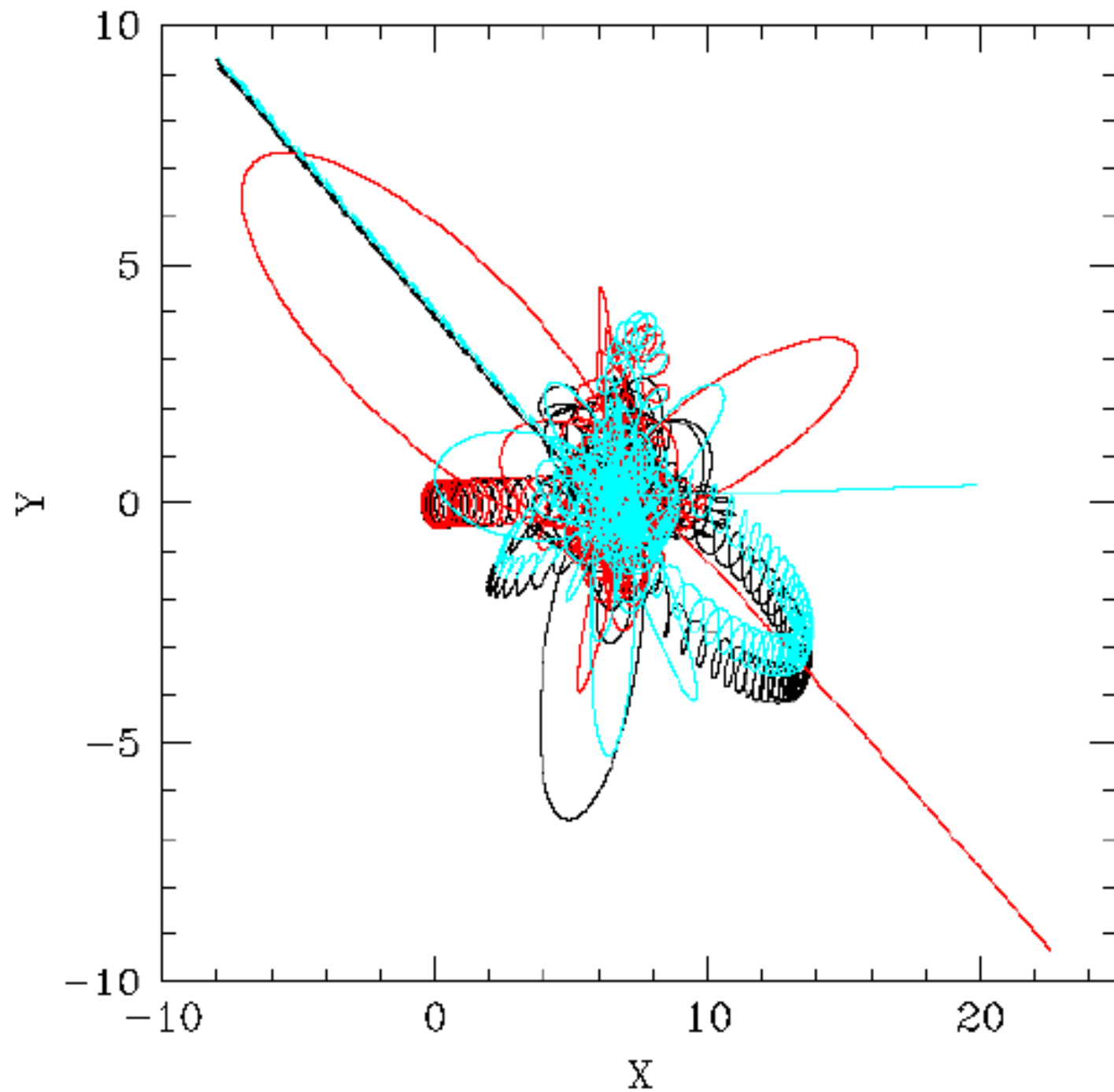
# EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

PROMPT  
EXCHANGE:



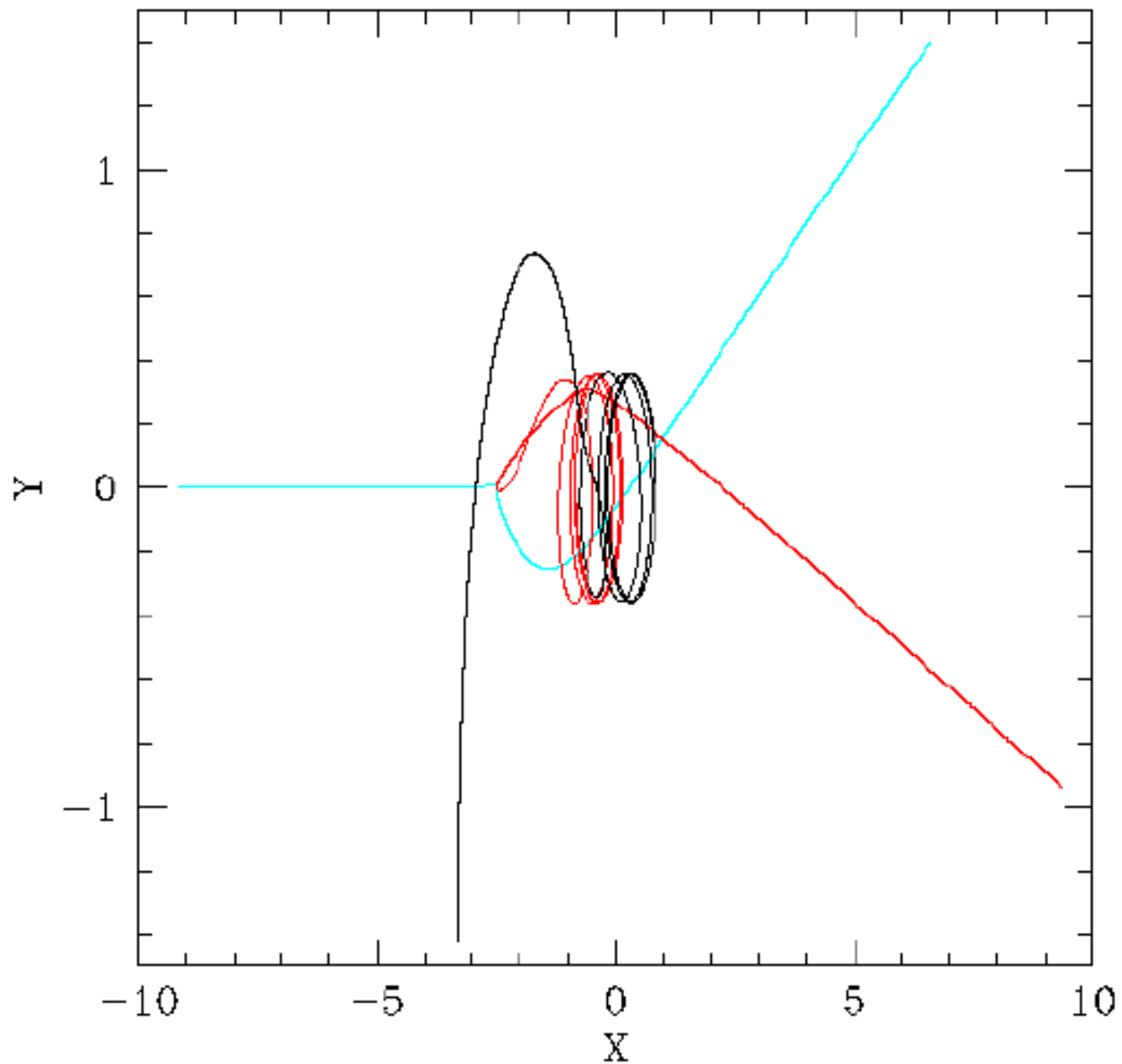
# EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

RESONANT  
EXCHANGE:



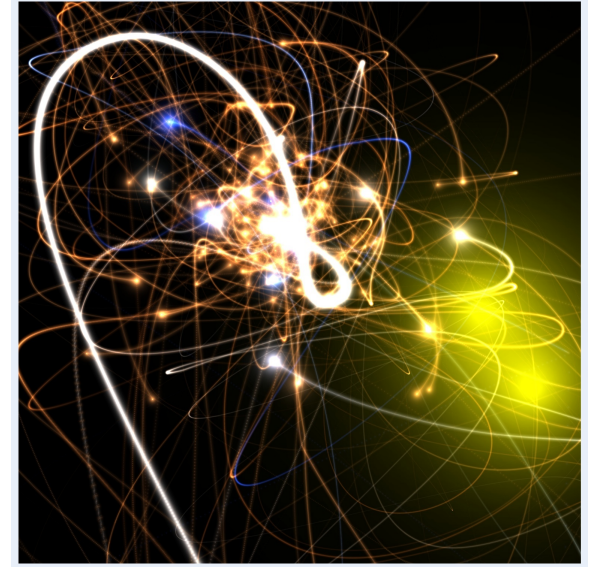
# EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

IONIZATION:



# EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

Credits: Aaron Geller (@Northwestern):



**Movie 2 :** binary – single interaction

[ciera.northwestern.edu/Research/visualizations/videos/Binary+single.mp4](http://ciera.northwestern.edu/Research/visualizations/videos/Binary+single.mp4)

**Movie 3 :** dynamical exchange

[ciera.northwestern.edu/Research/visualizations/videos/Binary+singleex.mp4](http://ciera.northwestern.edu/Research/visualizations/videos/Binary+singleex.mp4)

**Movie 4:** 5-body interaction (leads to a COLLISION!)

[ciera.northwestern.edu/Research/visualizations/videos/Triple+binary.mp4](http://ciera.northwestern.edu/Research/visualizations/videos/Triple+binary.mp4)

# Can we understand whether a binary will lose or acquire $E_b$ ?

YES, but ONLY in a STATISTICAL SENSE

We define **HARD BINARIES**: binaries with binding energy higher than the average kinetic energy of a star in the cluster

$$\frac{G m_1 m_2}{2 a} > \frac{1}{2} \langle m \rangle \sigma^2$$

$$\frac{G m_1 m_2}{2 a} < \frac{1}{2} \langle m \rangle \sigma^2$$

**SOFT BINARIES**: binaries with binding energy lower than the average kinetic energy of a star in the cluster

## **HEGGIE'S LAW (1975):**

Hard binaries tend to become harder (i.e. increase  $E_b$ )

Soft binaries tend to become softer (i.e. decrease  $E_b$ )

as effect of three-body encounters

## Cross Section for 3-body encounters:

Importance of binaries for dynamical encounters also due to the **LARGER CROSS SECTION** with respect to single stars

Simplest formalism for the cross section of stars and binaries:

### GEOMETRICAL CROSS SECTION

$$\Sigma_{GE} = \pi a^2$$

$$\Sigma_{GE} = \pi R_*^2$$

For binaries (scales with  $a^2$ )

For stars (scales with star radius  $R_*$ )

$$a > 10^{13} \text{ cm}, R_* \sim 10^{10-13} \text{ cm} \rightarrow a \gg R_*$$

The difference is even larger if we take a more realistic definition of **CROSS SECTION for 3-BODY ENCOUNTERS:**

$$\Sigma = \pi b_{max}^2$$

$b_{max}$  is the maximum impact parameter for a non-zero energy exchange between star and binary



# Cross Section for 3-body encounters:

How do we estimate  $b_{max}$ ?

## GRAVITATIONAL FOCUSING:

If the binary is significantly more massive than the single star, the TRAJECTORY of the single star is deflected by the binary, when approaching the pericentre.

The link between the impact parameter  $b$  and the effective pericentre  $p$  can be derived by the conservation of energy and angular momentum in the system of the reduced particle (see demonstration in the next slide):

$$p = \frac{G m_T}{v_\infty^2} \left[ \sqrt{1 + \left( \frac{v_\infty^2}{G m_T} \right)^2 b^2} - 1 \right]$$

where  $m_T \equiv m_1 + m_2 + m_3$

*Note: in most calculations  $v_\infty$  and  $\sigma$  (velocity dispersion) will be used as synonymous*

$$\text{if } \frac{G m_T}{v_\infty^2 b} \gg 1 \Rightarrow p \sim b^2 \frac{v_\infty^2}{2 G m_T}$$

# DEMONSTRATION of GRAVITATIONAL FOCUSING FORMULA:

## 1) Energy conservation

$$\frac{1}{2} \frac{m_3 (m_1 + m_2)}{m_T} v_\infty^2 - \frac{G m_1 m_2}{a} - \frac{G m_3 (m_1 + m_2)}{D} = \frac{1}{2} \frac{m_3 (m_1 + m_2)}{m_T} v_p^2 - \frac{G m_1 m_2}{a} - \frac{G m_3 (m_1 + m_2)}{p}$$

$$\frac{1}{2} \frac{v_\infty^2}{m_T} = \frac{1}{2} \frac{v_p^2}{m_T} - \frac{G}{p}$$

## 2) Angular momentum conservation

$$\vec{J}_\infty = (m_1 m_2) \sqrt{\frac{a_{in} G}{m_1 + m_2}} \hat{z} + b v_\infty \left[ \frac{m_3 (m_1 + m_2)}{m_T} \right] \hat{z}'$$

↑ parallel to  $b$ 
↑ perpendicular to  $b$

$$\vec{J}_p = (m_1 m_2) \sqrt{\frac{a_{in} G}{m_1 + m_2}} \hat{z} + p v_p \left[ \frac{m_3 (m_1 + m_2)}{m_T} \right] \hat{z}'$$

$$\Rightarrow b v_\infty \sim p v_p$$

## DEMONSTRATION of GRAVITATIONAL FOCUSING FORMULA:

1) and 2) together: 
$$\frac{1}{2} \frac{v_{\infty}^2}{m_T} - \frac{1}{2} \left( \frac{b v_{\infty}}{p} \right)^2 \frac{1}{m_T} + \frac{G}{p} = 0$$

Multiplying and dividing by  $p^2$

$$\frac{1}{2} \frac{v_{\infty}^2}{m_T} p^2 + G p - \frac{1}{2} \frac{(b v_{\infty})^2}{m_T} = 0$$

Finally:

$$p = \frac{-G \pm \sqrt{G^2 + \frac{b^2 v_{\infty}^4}{m_T^2}}}{v_{\infty}^2 / m_T} = \frac{G m_T}{v_{\infty}^2} \left[ \sqrt{1 + \left( \frac{v_{\infty}^2}{G m_T} \right)^2 b^2} - 1 \right]$$

$$\text{if } \frac{G m_T}{v_{\infty}^2 b} \gg 1 \Rightarrow p \sim b^2 \frac{v_{\infty}^2}{2 G m_T}$$

## Cross Section for 3-body encounters:

We now express  $b_{max}$  in terms of the pericentre distance  $\rho$  to obtain a useful formalism for the 3-body cross section (especially for massive binaries).

$$\Sigma = \pi \left( \frac{2 G m_T}{v_\infty^2} \right) \rho_{max}$$

A good choice for  $\rho_{max}$  (if we are interested only in the most energetic encounters) is  $\rho_{max} = a$

$$\Sigma = 2 \pi G \frac{m_T a}{v_\infty^2} \quad (*)$$

## 3-body interaction rate:

As usual, an interaction rate has the form

$$R = \frac{dN}{dt} = n \Sigma v_{\infty}$$

where  $n$  is the local density of stars.

For the cross section in (\*), the rate becomes

$$R = 2 \pi G \frac{m_T n a}{v_{\infty}}$$

*Note:* the rate depends

- (1) on the total mass of the interacting objects (more massive objects interact more),
- (2) on the semi-major axis of the binary (wider binaries have a larger cross section),
- (3) on the local density (denser environments have higher interaction rate),
- (4) on the local velocity field (systems with smaller velocity dispersion have higher interaction rate).

## Energy exchanges:

Most general formalism (from Energy conservation):

$$-\frac{G m_1 m_2}{a_i} + \frac{1}{2} \frac{m_3 (m_1 + m_2)}{m_T} v_\infty^2 = -\frac{G m_a m_b}{a_f} + \frac{1}{2} \frac{m_e (m_a + m_b)}{m_T} v_{fin}^2 \quad (+)$$

$m_a$ ,  $m_b$  and  $m_e$  are the final mass of the primary binary member, the final mass of the secondary binary member and the final mass of the single star, respectively (these may be different from the initial ones in the case of an exchange).

Change in binding energy:

$$\Delta E_b = \frac{1}{2} G \left( \frac{m_a m_b}{a_f} - \frac{m_1 m_2}{a_i} \right)$$

$$\frac{\Delta E_b}{E_{b, in}} = \left( \frac{m_a m_b}{m_1 m_2} \frac{a_i}{a_f} - 1 \right)$$

If NO EXCHANGE ( $m_a = m_1$ ,  $m_b = m_2$ )

$$\frac{\Delta E_b}{E_{b, in}} = \left( \frac{a_i}{a_f} - 1 \right)$$

If  $a_i / a_f > 1$  the quantity  $\Delta E_b$  is transferred to the kinetic energy of the involved centres of mass

**SUPERELASTIC ENCOUNTERS:** kinetic energy increases after interaction, because the binary is source of additional energy

## Energy exchanges:

Energy exchanges can be approximately QUANTIFIED if

1) *binary is hard*

2)  *$p$  is  $\ll 2a$*

3) *mass of the single star is small respect to binary mass  
(exchanges are unlikely)*

If 1), 2) and 3) hold, simulations show that

$$\frac{\Delta E_b}{E_b} \propto \frac{m_3}{m_1 + m_2}$$

# Energy exchanges:

Hills (1983, AJ, 88, 1269) defines the post-encounter energy parameter

$$\xi \equiv \frac{m_1 + m_2}{m_3} \frac{\langle \Delta E_b \rangle}{E_b}$$

Where  $\langle \Delta E_b \rangle$  is the average binding energy variation per encounter.

$\xi$  can be extracted from N-body simulations:

\*depends slightly on binary eccentricity ( $\xi \sim 2$  if  $e=0$ ,  $\xi \sim 6$  if  $e=0.99$ )

\*depends slightly on binary mass ratio ( $\xi \sim 2$  if  $m_1=m_2$ ,  $\xi \sim 4$  if  $m_1/m_2=10-30$ )

\* depends strongly on impact parameter (a factor of  $>200$  between  $b=0$  and  $b=20a$ )

by averaging over relevant impact parameters  **$\xi = 0.2 - 1$**

From  $E_b$  definition

$$\langle \Delta E_b \rangle = \xi \frac{m_3}{m_1 + m_2} \frac{G m_1 m_2}{2 a}$$



## Hardening rate:

Rate of binding energy exchange for a hard binary

$$\frac{dE_b}{dt} = \langle \Delta E_b \rangle \frac{dN}{dt} = \xi \frac{m_3}{m_1 + m_2} E_b \frac{dN}{dt}$$

Where  $dN/dt$  is the collision rate. Using formalism in slide 20

$$\frac{dE_b}{dt} = \xi \frac{\langle m \rangle}{m_1 + m_2} E_b \frac{2 \pi G (m_1 + m_2) n a}{\sigma}$$

Average star mass (because average energy exchange)

Note:  $\langle m \rangle n = \rho$  (local mass density of stars)

$$\frac{dE_b}{dt} = \pi \xi G^2 \frac{\rho}{\sigma} m_1 m_2$$

*\*Depends only on cluster environment and binary mass!*

*\*Constant in time, if the cluster properties do not change and if the binary members do not exchange:*

→ *'A hard binary hardens at a constant rate'* (Heggie 1975, 3-body Bible)

## Hardening rate:

Expressing  $a$  in terms of  $E_b$  (assuming  $m_1$  and  $m_2$  constant, i.e. no exchange)

$$\frac{d}{dt} \left( \frac{1}{a} \right) = \frac{2}{G m_1 m_2} \frac{dE_b}{dt} = 2 \pi G \xi \frac{\rho}{\sigma}$$

Also called **HARDENING RATE**

From it we can derive the average time evolution of the semi-major axis of a hard binary:

$$\frac{da}{dt} = -2 \pi G \xi \frac{\rho}{\sigma} a^2$$

**It means that the smaller  $a$ , the more difficult is for the binary to shrink further (because cross section becomes smaller).**

When binary is very hard, three body encounters are no longer efficient: further evolution of the binary is affected by tidal forces or by gravitational wave emission.

## Hardening timescale:

**How would you define then  
the timescale for a binary  
to harden by three-body encounters?**

## Hardening timescale:

Timescale for a binary to harden by 3-body encounters

$$t_h = \left| \frac{a}{\dot{a}} \right| = \frac{1}{2\pi G \xi} \frac{\sigma}{\rho} \frac{1}{a}$$

Depends only on

- \* local velocity dispersion
- \* local mass density
  
- \* semi-major axis
  
- \* parameter for efficiency of energy exchange

# Other relevant timescales

## 1) INTERACTION TIMESCALE (from interaction rate)

$$t_{3b} = \frac{\sigma}{2 \pi G m_T n a}$$

## 2) DYNAMICAL FRICTION TIMESCALE

$$t_{df} = \frac{3}{4 (2 \pi)^{1/2} G^2 \ln \Lambda} \frac{\sigma^3}{m_T \rho(r)}$$

If  $t_{df} \ll t_{3b}$ , dynamical friction washes velocity changes induced by 3-body encounters (especially for very small  $a$ ) and especially out of core (where  $\sigma$  drops)

If  $t_{df} \gg t_{3b}$ , 3-body encounters dominate the binary velocity (especially for large  $a$ )

## Recoil velocities:

Most general expression of recoil velocity for the reduced particle (Sigurdsson & Phinney 1993)

$$v_{fin} = \sqrt{\frac{m_3 (m_1 + m_2)}{m_e (m_a + m_b)} v_\infty^2 + \frac{2 m_T}{m_e (m_a + m_b)} \Delta E_b}$$

$m_a$ ,  $m_b$  and  $m_e$  are the final mass of the primary binary member, the final mass of the secondary binary member and the final mass of the single star, respectively (these may be different from the initial ones in the case of an exchange).

$$\frac{1}{2} \frac{m_3 (m_1 + m_2)}{m_T} v_\infty^2 + \Delta E_b = \frac{1}{2} \frac{m_e (m_a + m_b)}{m_T} v_{fin}^2$$

*What happens to the binary, then?*

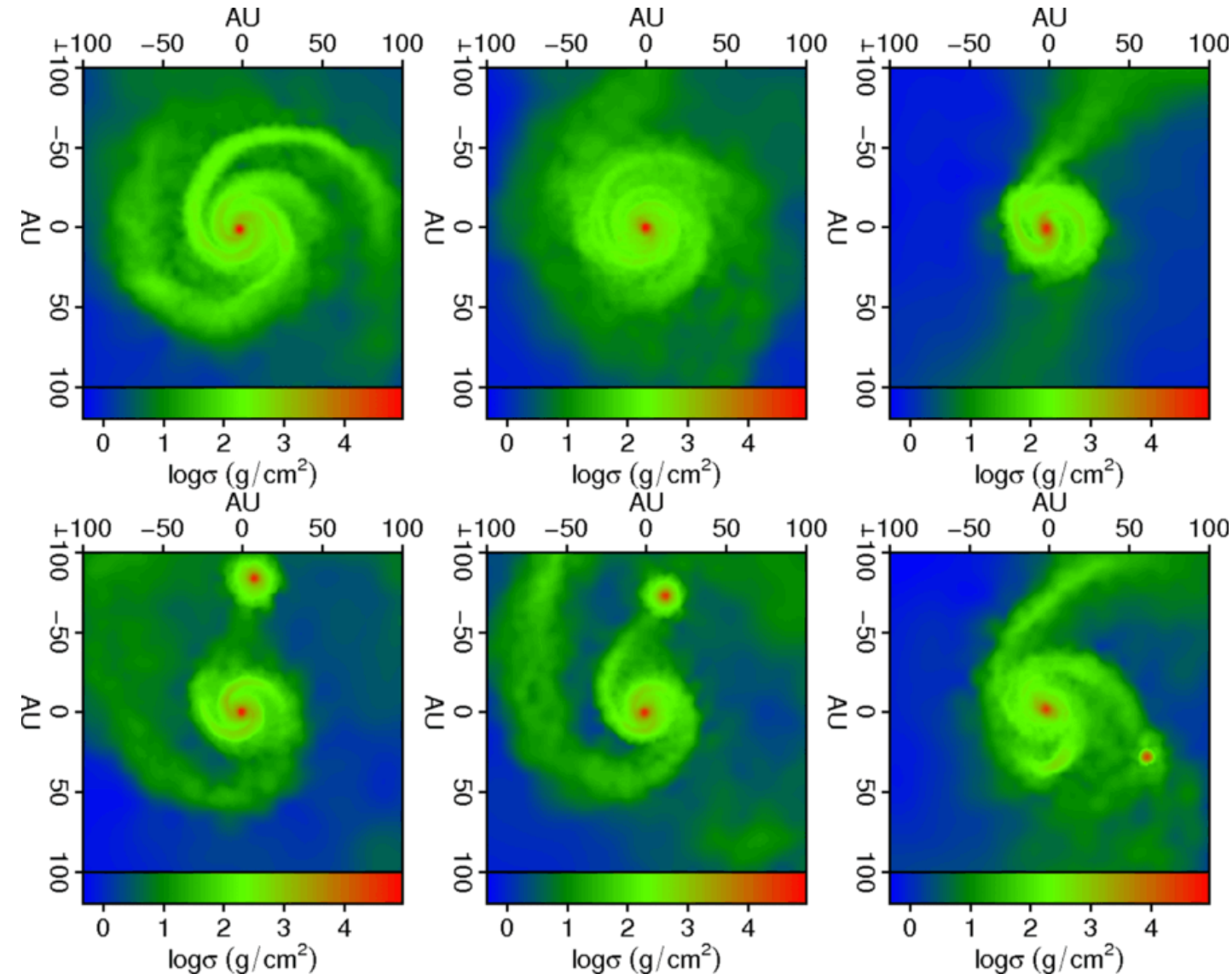
The recoil of the binary (if the binary is more massive than the single star -i.e. the motion of the single star coincides almost with that of the reduced particles) follows from conservation of linear momentum

$$v_{rec} = \frac{m_e}{m_T} v_{fin}$$

# Origin of binaries:

Which are the formation pathways of binaries?

**1) primordial binaries:** binaries form from the same accreting clump in the parent molecular cloud (very difficult to understand with simulations)

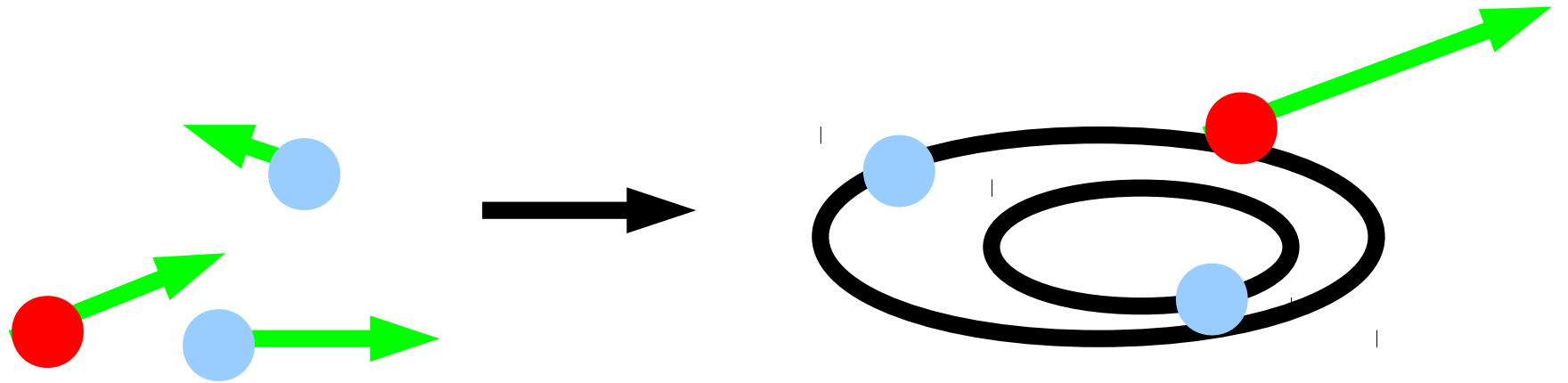


SPH simulation of a single star and a binary formed from a molecular cloud

Hayfield et al. 2011, MNRAS, 2011, 417, 1839

# Origin of binaries:

**2) three-body induced binaries:** 3 single stars pass close to each other and one of the three brings away sufficient energy to leave the others bound. Unlikely unless high density (core collapse)



$$\frac{dn_b}{dt} = 1.91 \times 10^{-13} \left( \frac{n}{10^4 \text{ pc}^3} \right)^3 \left( \frac{m}{m_\odot} \right)^5 \left( \frac{10 \text{ km s}^{-1}}{\langle v \rangle} \right)^9 \text{ pc}^{-3} \text{ yr}^{-1}$$



# Origin of binaries:

- 3) tidally induced binaries:** two stars pass very close to each other so that they feel each other tidal field  
→ energy dissipation produces a bound couple

Unlikely unless high density (core collapse)

*Tidal radius (from Roche limit):*

$$r_t = r_2 \left( \frac{2 m_1}{m_2} \right)^{1/3}$$

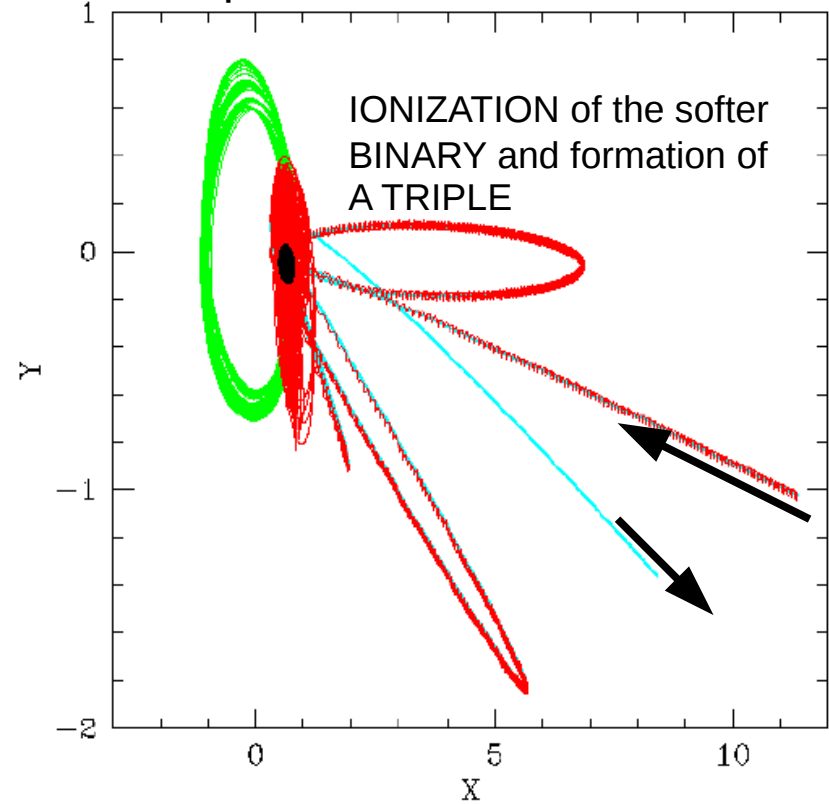
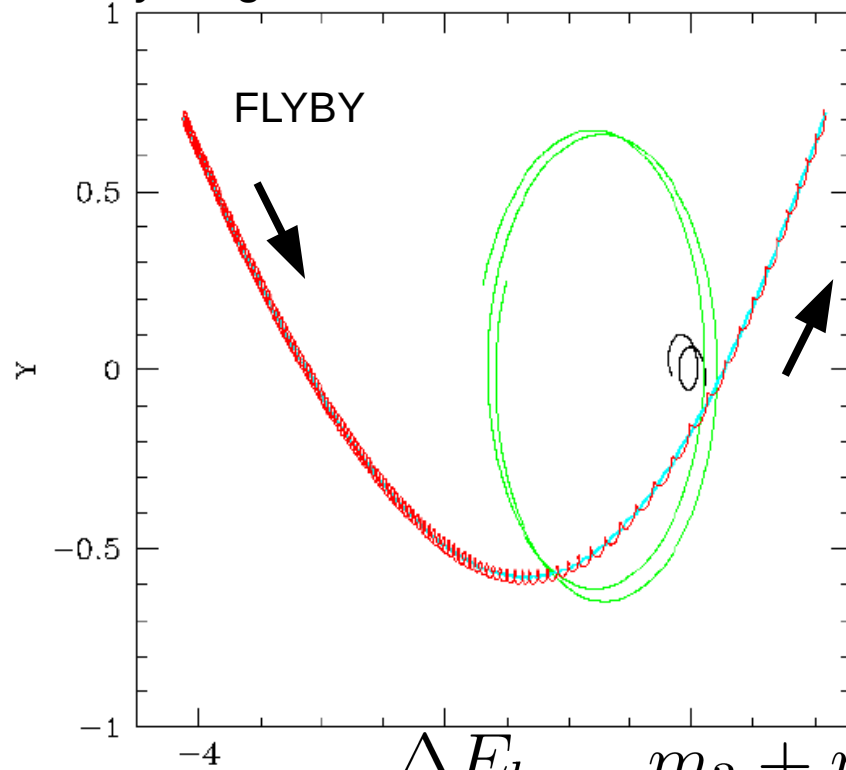
valid if  $m_1 > m_2$

- 4) binaries formed by exchange:**

in general, they are not considered new binaries as they come from a pre-existing binary

## 4- (5-, 6-, ...) body encounters:

Everything we said still holds, but with higher level of complications



Energy exchanges:  $\frac{\Delta E_b}{E_b} \propto \frac{m_3 + m_4}{m_1 + m_2}$

Critical velocity for ioniz.  $v_c = \sqrt{\frac{G m_T}{(m_1 + m_2)(m_3 + m_4)} \left( \frac{m_1 m_2}{a_1} + \frac{m_3 m_4}{a_2} \right)}$

Even RARE STABLE TRIPLES can form  
*(Mardling & Aarseth 1999 stability criterion)*  
 $r_p, e_{ou}$  = pericentre and eccentricity of the outer binary  
 $a_{in}$  = semi-major axis of the inner binary  
 $q = m_{ou}/m_{in}$  mass ratio (outer to inner binary)

$$\frac{r_p}{a_{in}} \geq 2.8 \left[ (1 + q) \frac{1 + e_{ou}}{\sqrt{1 - e_{ou}^2}} \right]^{2/5}$$

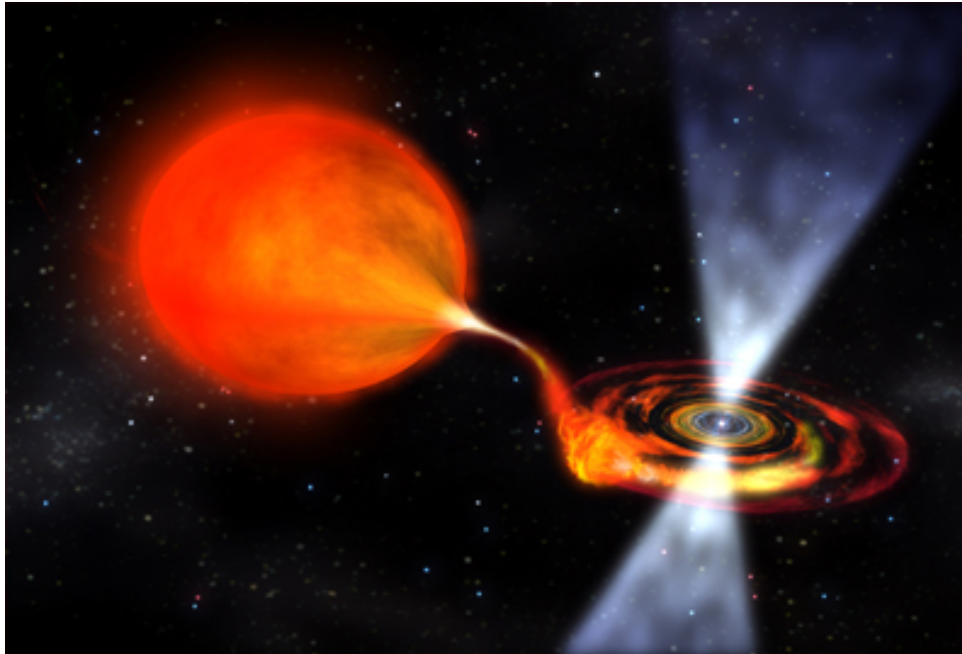
# Role of binaries in cluster evolution:

- 1\* Density increase during core collapse HARDENS primordial binaries and ENHANCES FORMATION of binaries by encounters of three single stars and by tidal captures
- 2\* HARD primordial binaries and new 3-body formed binaries REVERSE core collapse by transferring their internal energy to  $K$  of stars in core VIA 3-BODY ENCOUNTERS
- 3\* stars that undergo 3-body encounters with hard binaries are ejected ( $|W|$  decreases) or remain in the core and transfer  $K$  to other stars
- 4\* *Note: HARD binaries transfer  $K$  to the core but SOFT binaries extract  $K$  from the core. Why are hard binaries predominant?  
TOTAL  $|E_i|$  of HARD BINARIES  $\gg$  TOTAL  $|E_i|$  of SOFT BINARIES!!!!*
- 5\* Mass losses by stellar winds and supernovae can help, but only if the timescale for massive star evolution is  $\sim$  core collapse time!!! (see last lecture)
- 6\* If there are  $>1$  very hard binaries might eject each other by 4-body encounters

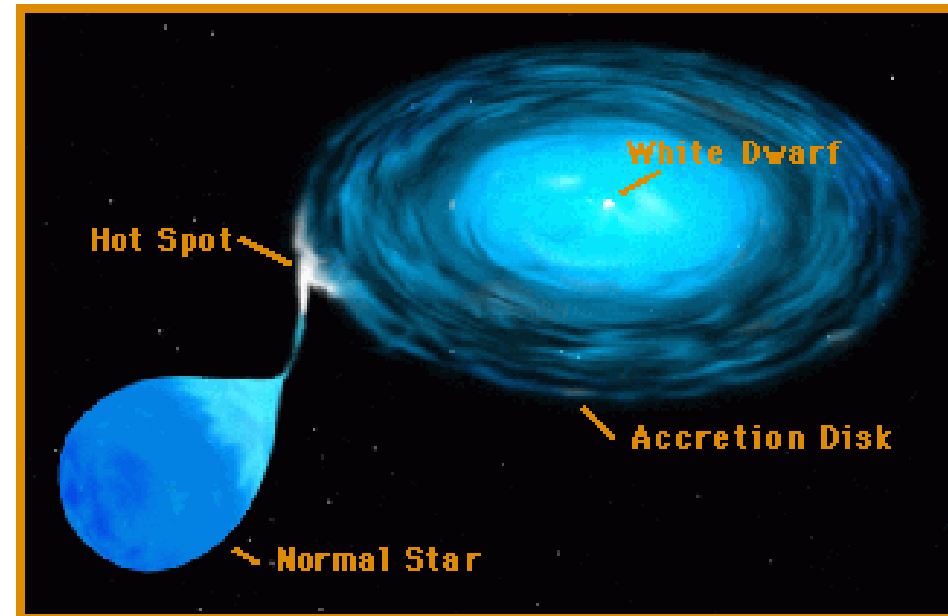
# Role of binaries in formation of exotica:

Binaries and three-body encounters are the main suspects for the formation of STELLAR EXOTICA, such as

- \* blue straggler stars
- \* massive BHs and intermediate-mass BHs
- \* millisecond pulsars



- \* cataclysmic variables



## References:

- \* **Mapelli M., Master thesis, 2003, in Italian,  
<http://web.pd.astro.it/mapelli/images/tesi.ps.gz>**
- \* Binney & Tremaine, Galactic Dynamics, First edition, 1987, Princeton University Press
- \* Hut & Bahcall, Binary-single star scattering. I - Numerical experiments for equal masses, 1983, ApJ, 268, 319
- \* Sigurdsson S., Phinney S., Binary-single star interactions in globular clusters, 1993, ApJ, 415, 631
- \* Davies M. B., Stellar exotica produced from stellar encounters, 2002, ASP
- \* Spitzer L., Dynamical evolution of globular clusters, 1987, Princeton University Press
- \* Elson R. & Hut P., Dynamical evolution of globular clusters, Ann. Rev. Astron. Astrophys., 1987, 25, 565