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Dynamics of Stars and Black Holes in Dense Stellar Systems:

Lecture I:

STAR CLUSTERS AS COLLISIONAL SYSTEMS

1. Relevant timescales of dense stellar systems
2. Early evolution of dense star systems
3. Equilibrium models
4. N-body simulations of star clusters + star evolution

Dynamics of Stars and Black Holes in Dense Stellar Systems:

Slides at

http://web.pd.astro.it/mapelli/lectures.html

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1. Relevant timescales of dense stellar systems:

- what is the utility of a timescale?

- what is two-body relaxation and its timescale?

- what is dynamical friction and its timescale?

NOTE: the blue box in these slides will be a question for you

1. Relevant timescales of dense stellar systems:

Timescale:

* characteristic time for a process to take place

* the shorter it is, the more efficient the process

* comparing timescales we can understand what will occur in a system

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TWO-BODY RELAXATION TIMESCALE:

Time for a star to lose memory of its initial velocity completely, as an effect of gravitational encounters

AKA

Time needed for a system to relax (=reach relaxation) by two-body encounters

1. **Relevant timescales of dense stellar systems:**

COLLISIONAL/COLLISIONLESS?

Collisional systems are systems where interactions between particles are EFFICIENT with respect to the lifetime of the system

Collisionless systems are systems where interactions are negligible

When is a system collisional/collisionless?

When its RELAXATION TIMESCALE is short wrt lifetime

Gravity is a LONG-RANGE force → cumulative influence on each star/body of distant stars/bodies is important: often more important than influence of close stars/bodies

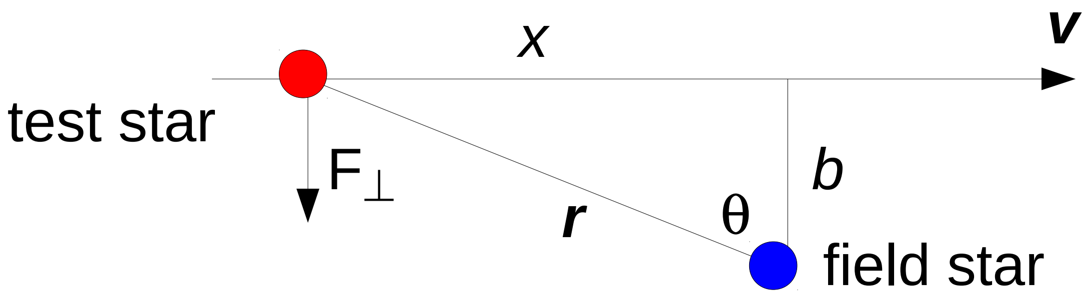
Let us consider a IDEALIZED galaxy of N identical stars with mass *m*, size *R* and uniform density
Let us focus on a single star that crosses the system

How long does it take for this star to change its initial velocity completely?,

$$\frac{\delta\vec{v}_\perp}{\vec{v}} \sim 1$$

where $\delta v_\perp =$ component of velocity perpendicular to the initial velocity vector

Let us assume that our test star passes close to a field star at relative velocity *v* and impact parameter *b*



The test star and the perturber interact with a force

$$F_\perp = \frac{G\,m^2}{r^3}\,r_\perp = \frac{G\,m^2}{r^2}\,\cos\theta$$

$$= \frac{G\,m^2\,b}{(x^2+b^2)^{3/2}} = \frac{G\,m^2}{b^2}\,\frac{1}{\left[1+\left(\frac{v\,t}{b}\right)^2\right]^{3/2}}$$

1. Relevant timescales of dense stellar systems:

From Newton's second law $m\,\dot{v}_\perp = F_\perp$

we get that the perturbation of the velocity integrated over one entire encounter is

$$\delta v_\perp = \int_{-\infty}^{+\infty} \frac{F_\perp}{m} dt = \frac{G\,m}{b^2} \int_{-\infty}^{+\infty} \frac{dt}{\left[1 + \left(\frac{v\,t}{b}\right)^2\right]^{3/2}}$$

$$= \frac{G\,m}{b\,v} \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^{3/2}} = \frac{2\,G\,m}{b\,v}$$

$$dt = \frac{b}{v} d\left(\frac{v\,t}{b}\right)$$

Accel. at closest approach | Force duration

$$2\left(\frac{G\,m}{b^2}\right)\left(\frac{b}{v}\right)$$

Now we account for all the particles in the system

Surface density of stars in idealized galaxy: $$\frac{N}{\pi R^2}$$

Number of interactions per unit element:

$$\delta n = \frac{N}{\pi R^2} 2 \pi b \, db$$

We define

$$\delta v^2_{\rm TOT} = \delta n \, \delta v^2_{\perp} = \frac{2N}{R^2} \left(\frac{2Gm}{bv} \right)^2 b \, db$$

And we integrate over all the possible impact parameters...

And we integrate over all the possible impact parameters...

$$\delta v_{\mathrm{TOT}}^2 = 8\,N\,\left(\frac{G\,m}{R\,v}\right)^2 \int_{b_{\mathrm{min}}}^{R} \frac{db}{b}$$

$$\boxed{\delta v_{\mathrm{TOT}}^2 = 8\,N\,\left(\frac{G\,m}{R\,v}\right)^2 \ln\frac{R}{b_{\mathrm{min}}}}$$

* low integration limit: smallest *b* to avoid close encounter $\delta v_{\perp} \sim v$

$$v = \frac{2\,G\,m}{b_{\mathrm{min}}\,v} \quad \Longrightarrow b_{\mathrm{min}} = \frac{2\,G\,m}{v^2}$$

* top integration limit: size *R* of the system

1. Relevant timescales of dense stellar systems:

$$\delta v^2_{\rm TOT} = 8\,N\,\left(\frac{G\,m}{R\,v}\right)^2\,\ln\left(\frac{R\,v^2}{2\,G\,m}\right)$$

Typical speed of a star in a virialized system

$$N\,m\,v^2 = \frac{G\,(N\,m)^2}{R} \qquad \Longrightarrow v^2 = \frac{G}{N\,m}R$$

Replacing *v*

$$\frac{\delta v^2_{\rm TOT}}{v^2} = \frac{8\,\ln N}{N}$$

1. Relevant timescales of dense stellar systems:

$$v^2 = \frac{N}{8 \ln N} \, \delta v^2_{\rm TOT}$$

Number of crossings of the system for which $\frac{\delta v^2_{\rm TOT}}{v^2} = 1$

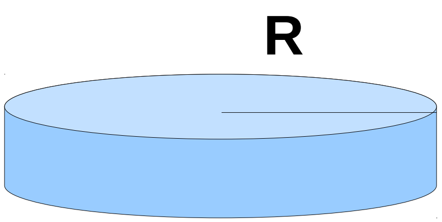
$$n_{\rm cross} \, \delta v^2_{\rm TOT} = v^2 = \frac{N}{8 \ln N} \, \delta v^2_{\rm TOT}$$

1. Relevant timescales of dense stellar systems:

Number of crossings of the system for which $\frac{\delta v_{\rm TOT}^2}{v^2} = 1$

$$n_{\rm cross} = \frac{N}{8\,\ln N}$$

CROSSING TIME = time needed to cross the system (also named DYNAMICAL TIME)



$$t_{cross} = \frac{R}{v}$$

$$= \sqrt{\frac{R^3}{G\,M}} = \frac{1}{\sqrt{G\,\rho}}$$

1. Relevant timescales of dense stellar systems:

RELAXATION TIME = time necessary for stars in a system to lose completely the memory of their initial velocity

$$t_{\rm rlx} = n_{\rm cross}\, t_{\rm cross} = \frac{N}{8\,\ln N}\,\frac{R}{v}$$

with more accurate calculations, based on diffusion coefficients (Spitzer & Hart 1971):

$$t_{\rm rlx} = 0.34\,\frac{\sigma^3(r)}{G^2\,m_*\,\rho_*(r)\,\ln\Lambda}$$

The two expressions are almost equivalent

If we put $\sigma = v = (G\, N\, m \,/\, R)^{1/2}$
and ρ μ $N\, m/ R^3$
and $\ln\Lambda \sim \ln N$

$$t_{\rm rlx} \propto \frac{\sigma^3}{G^2\, m\, \rho \ln\Lambda} \sim \frac{(G\, N\, m\, R^{-1})^{3/2}}{G^2\, N\, m^2\, R^{-3} \ln N}$$

$$\sim G^{-1/2}\, N^{1/2}\, m^{-1/2}\, R^{3/2} \ln N^{-1}\, \frac{v}{v}$$

multiply and divide per *v*

$$\sim \left(\frac{N\, R}{G\, m}\right)^{1/2} v \ln N^{-1}\, \frac{R}{v}$$

Rearrange and substitute again *v*

$$\sim \frac{N}{\ln N}\, \frac{R}{v}$$

1. Relevant timescales of dense stellar systems:

USEFUL BACK-OF-THE-ENVELOPE expression for t_rlx:

If we put $v = (G\, N\, m\, /\, R)^{1/2}$

$$t_{rlx} \sim \frac{N}{\ln N}\,\frac{R}{v} \qquad \sim \frac{N}{\ln N}\,\frac{R^{3/2}}{(G\,N\,m)^{1/2}}$$

$$\sim \frac{N^{1/2}\,R^{3/2}}{(G\,m)^{1/2}\,\ln N} \qquad \sim \frac{(m\,N)^{1/2}\,R^{3/2}}{G^{1/2}\,m\,\ln N}$$

$$\sim 15\,\mathrm{Myr}\,\left(\frac{M_{TOT}}{10^4\,M_\odot}\right)^{1/2}\,\left(\frac{R}{1\,\mathrm{pc}}\right)^{3/2}\,\left(\frac{1\,M_\odot}{m}\right)$$

Portegies Zwart 2006

RELAXATION & THERMALIZATION

*Relaxation and thermalization are almost **SYNONYMOUS**!*

* ***Thermalization**:*

- *is one case of relaxation*
- *is defined for gas (because needs definition of **T**), but can be used also for stellar system (kinetic extension of T)*
- *is the **process of particles reaching <u>thermal equilibrium</u> through mutual interactions** (involves concepts of equipartition and evolution towards maximum entropy state)*
- *has velocity distribution function: **Maxwellian** velocity*

**** Relaxation**:*

- *is defined not only for gas*
- *is the **process of particles reaching equilibrium through mutual interactions***
  (but there might be many processes driving to relaxation not only 2-body relaxation)

Which is the typical t_{rlx} of stellar systems?

EXERCISE: CALCULATE t_{rlx} for

- **the disk of the Milky Way**
- **the core of a globular cluster**

Which is the typical t_{rlx} of stellar systems?

* **Globular clusters, dense young star clusters, nuclear star clusters** (far from SMBH influence radius)

R~1-10 pc, N~10$^{3-6}$ stars, v~1-10 km/s

$$t_{rlx} \sim 10^{7-10}\ \mathrm{yr}$$

→ **COLLISIONAL**

* **Galaxy field/discs**

R~10 kpc, N~10^{10} stars, v~100-500 km/s

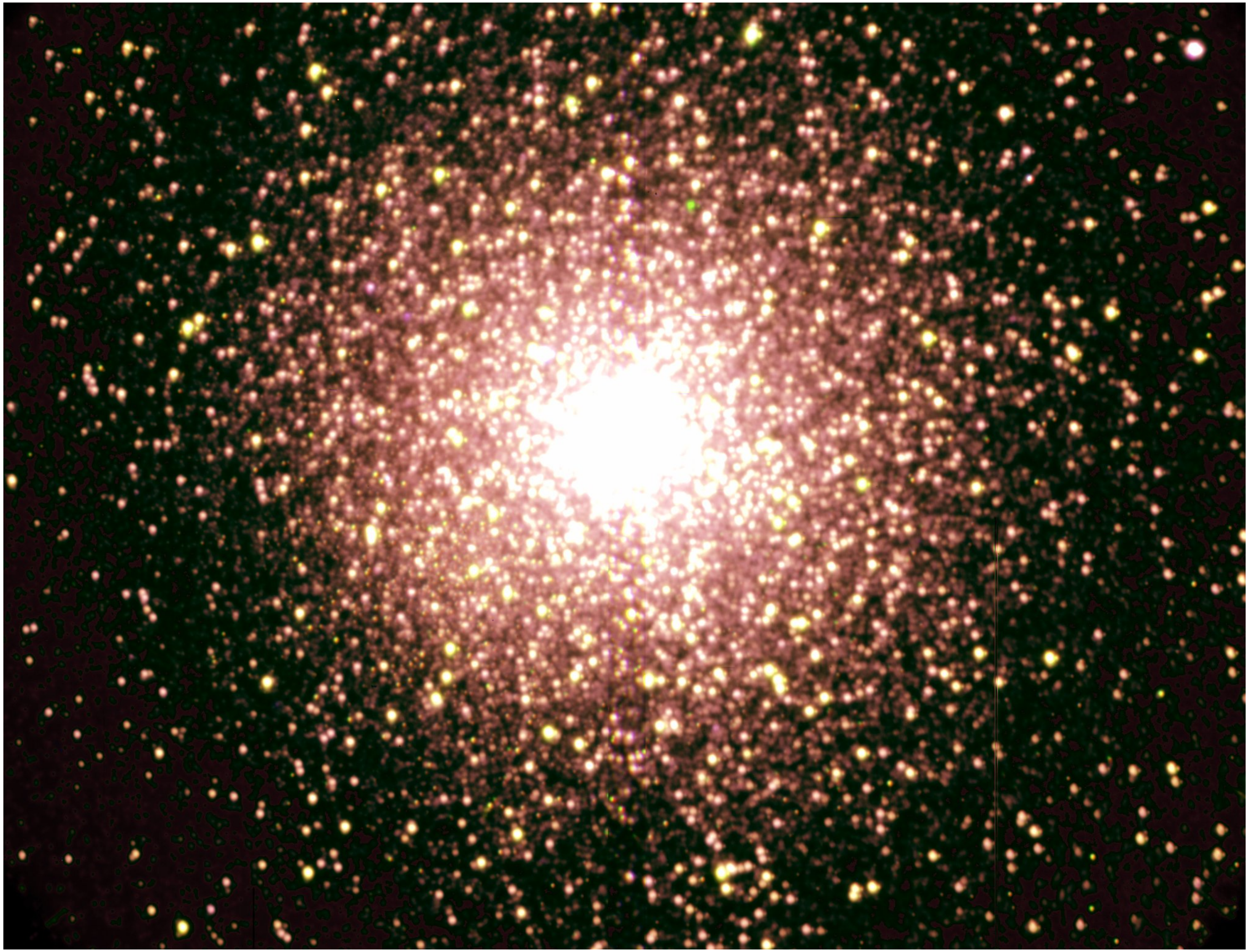
$$t_{rlx} \gg \text{Hubble time}$$

→ **COLLISIONLESS**

described by collisionless Boltzmann equation

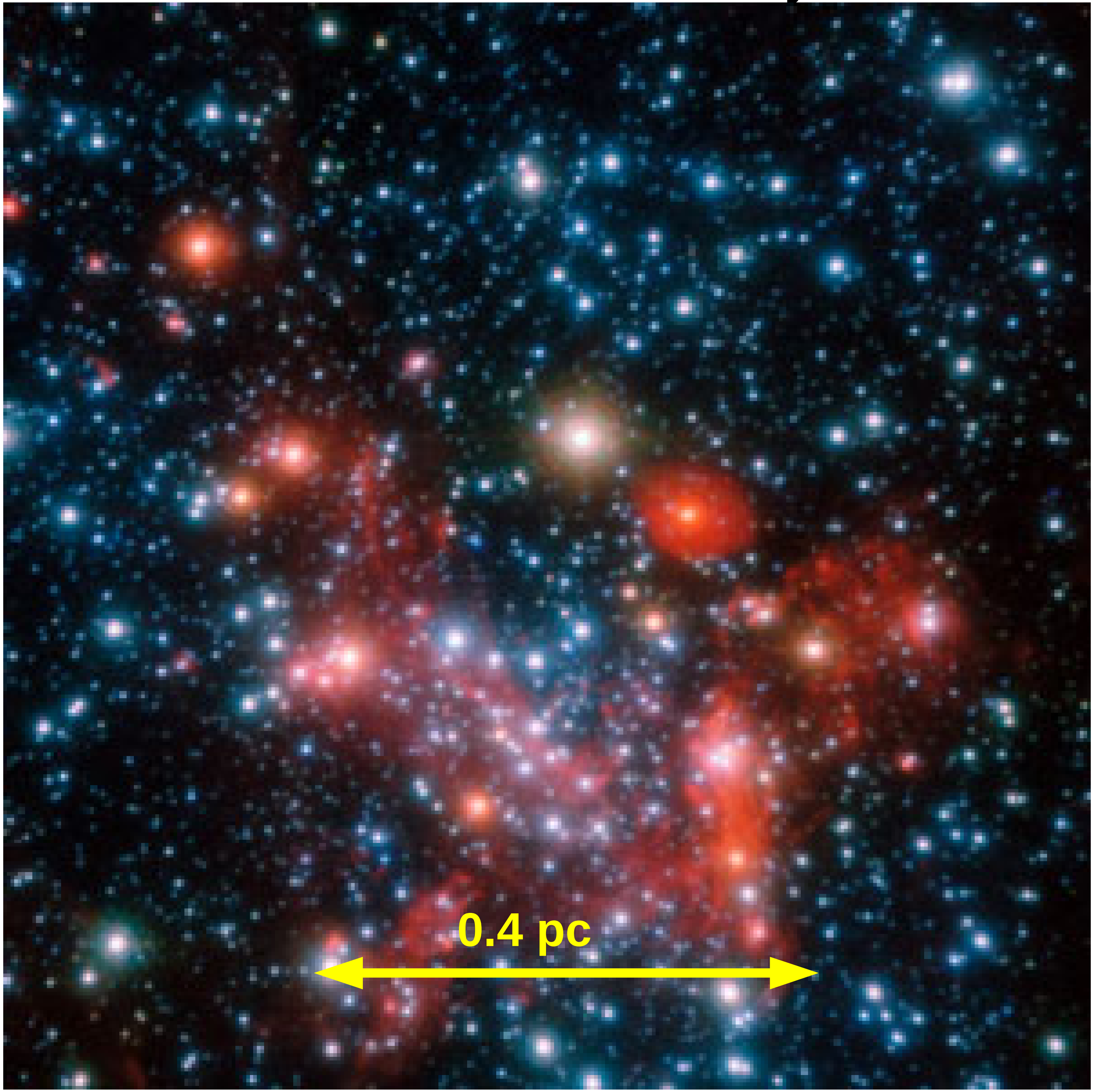
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1,6} \frac{\partial x_i}{\partial t} \frac{\partial f}{\partial x_i} = 0$$

EXAMPLES of COLLISIONAL stellar systems



Globular clusters (47Tuc), by definition

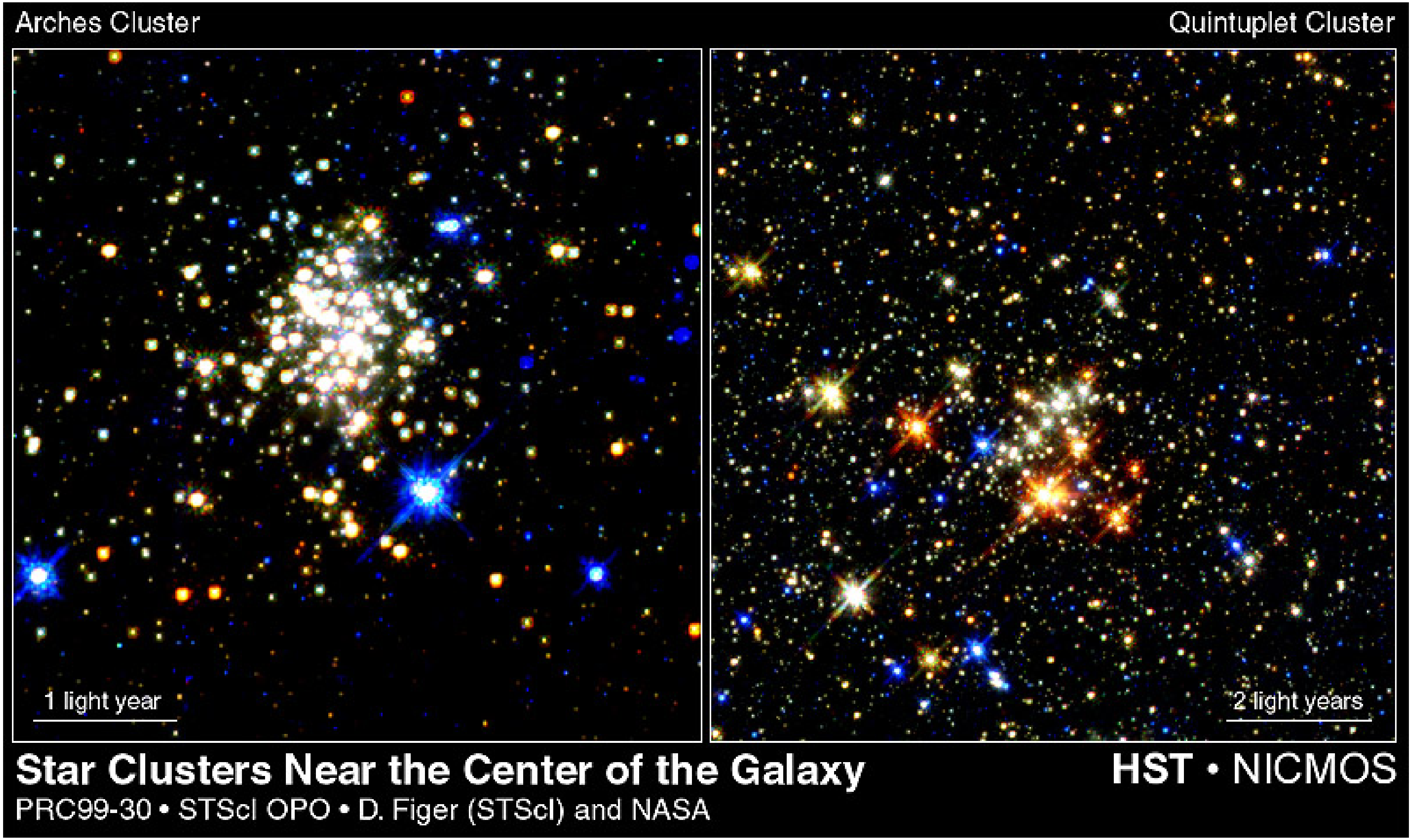
EXAMPLES of COLLISIONAL stellar systems



Nuclear star clusters (MW)
NaCo @ VLT
Genzel+2003

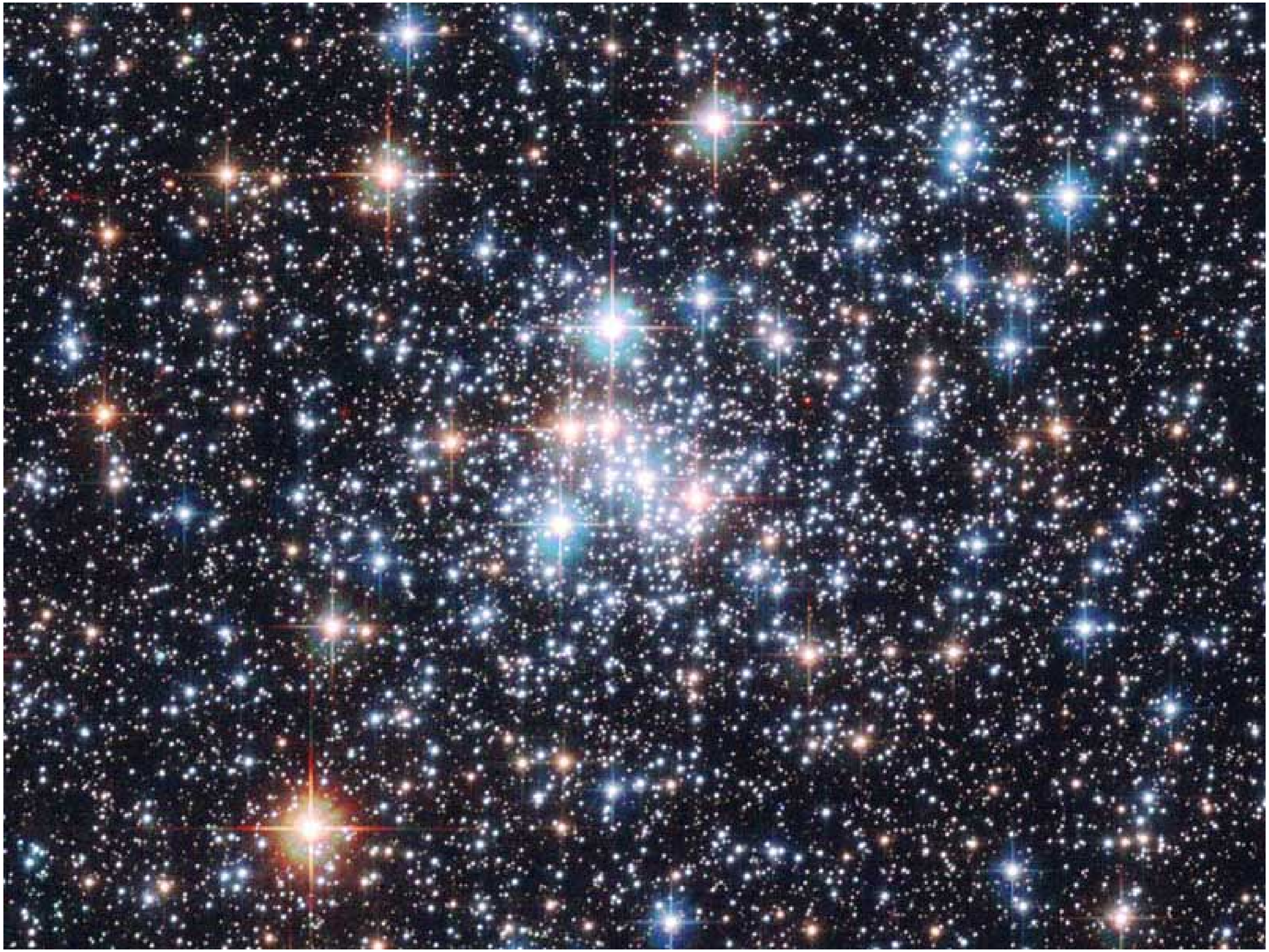
1. Relevant timescales of dense stellar systems:

EXAMPLES of COLLISIONAL stellar systems



Young dense star clusters (Arches, Quintuplet)

EXAMPLES of COLLISIONAL stellar systems



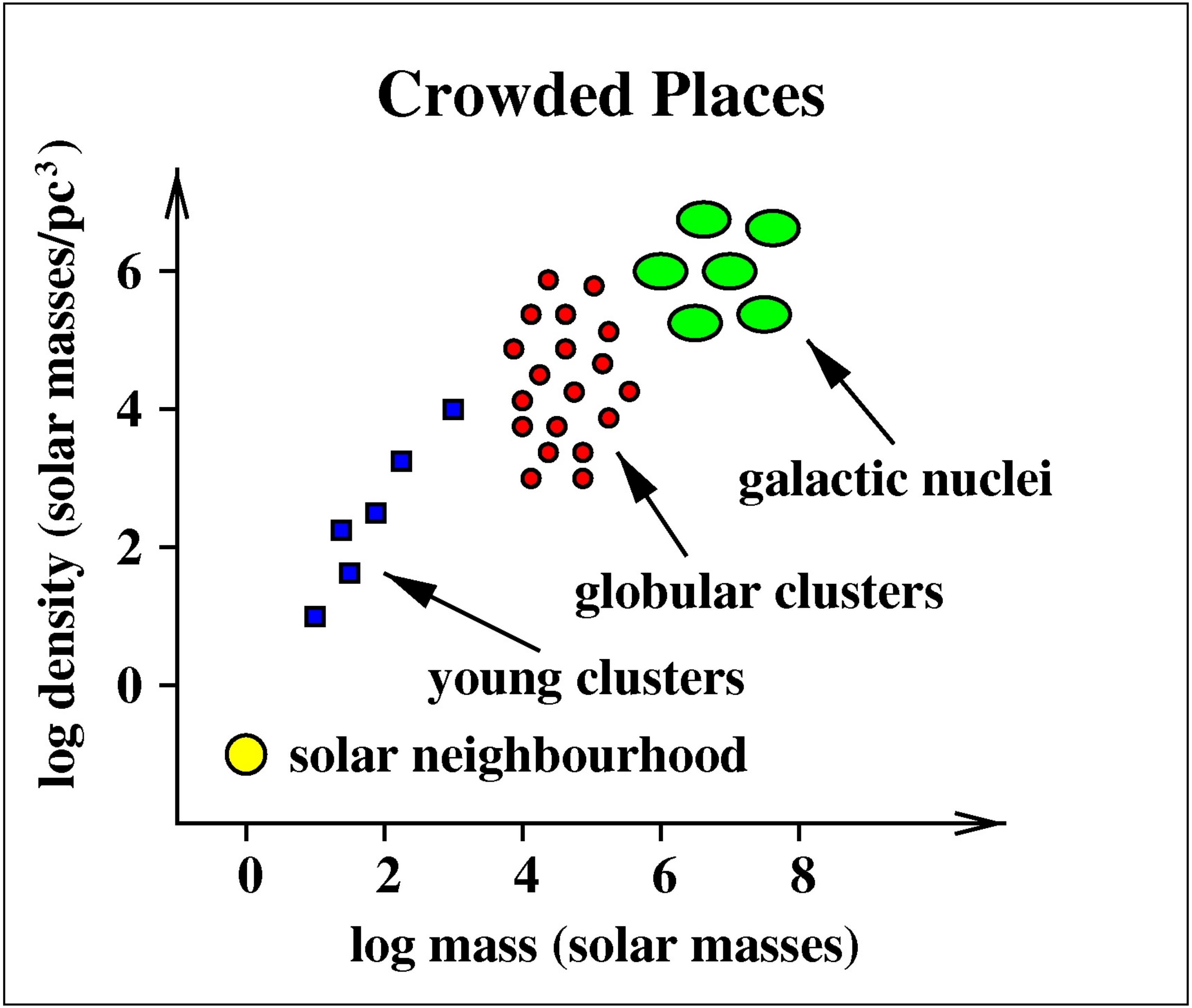
Open clusters, especially in the past (NGC290)

EXAMPLES of COLLISIONAL stellar systems

Embedded clusters, i.e. baby clusters (RCW 38)
NaCo @ VLT

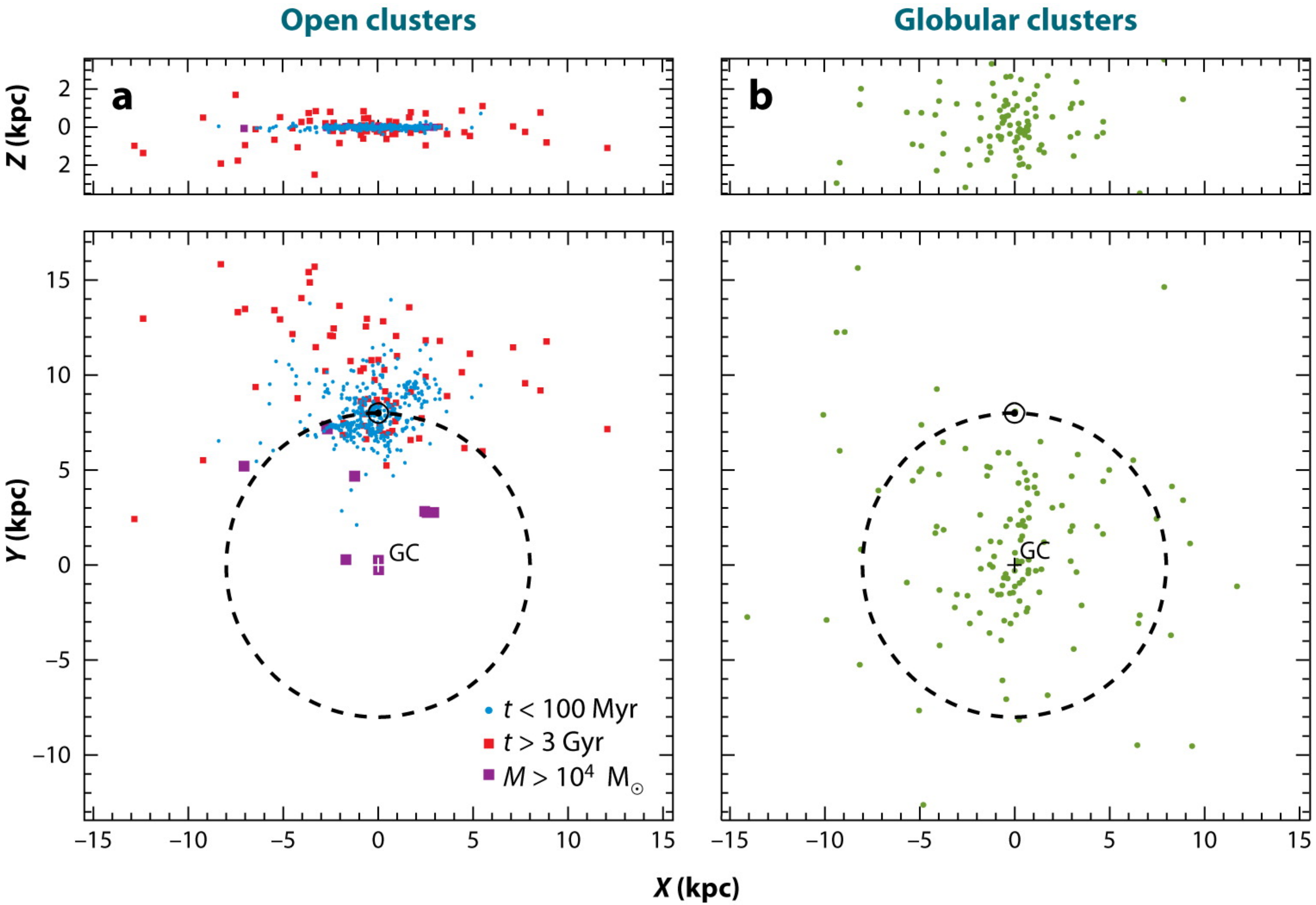
1. Relevant timescales of dense stellar systems:

DENSITY & MASS ORDER OF MAGNITUDES



M. B. Davies, 2002, astroph/0110466

DISTRIBUTION of COLLISIONAL stellar systems in the MILKY WAY

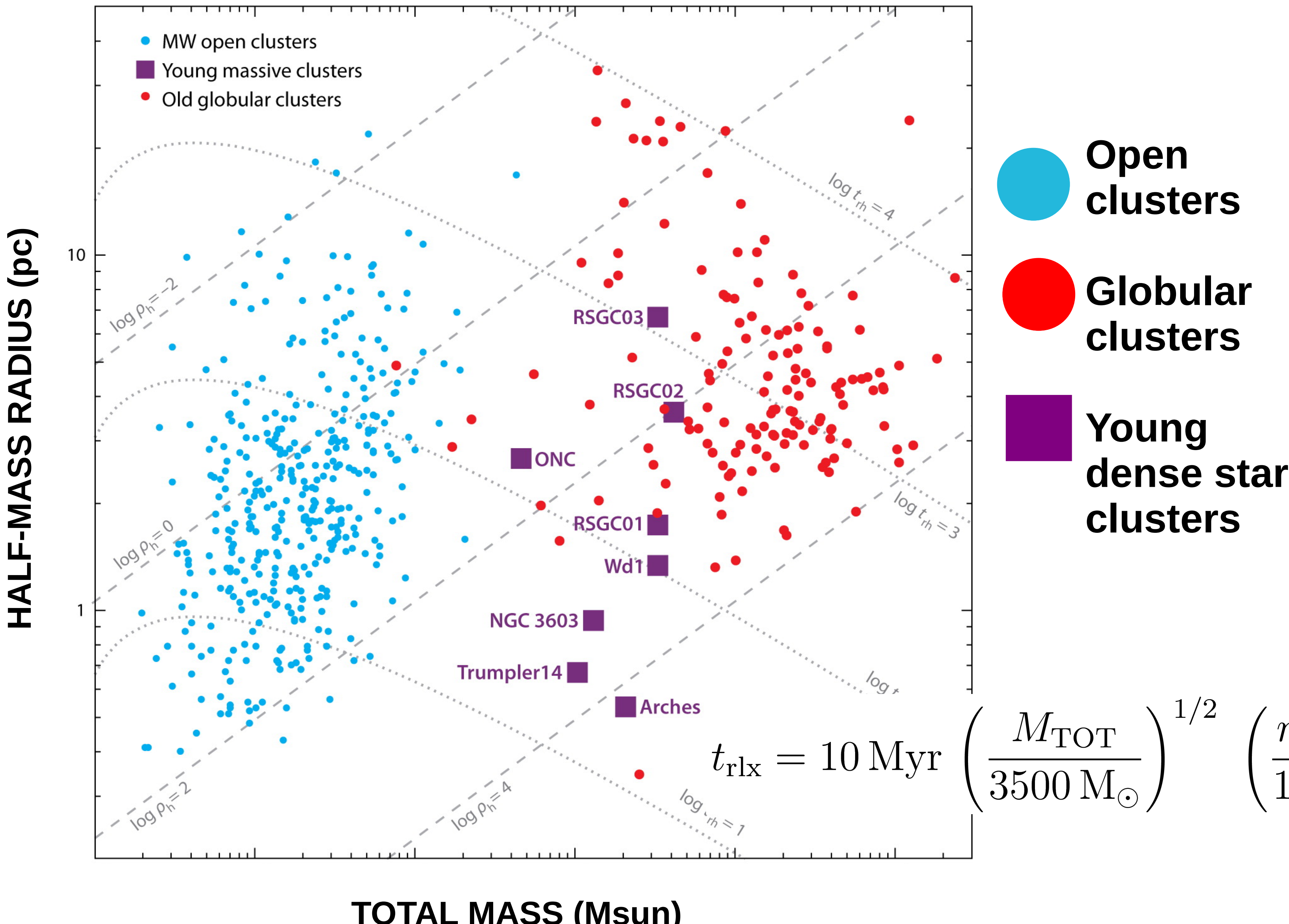


GLOBULAR CLUSTERS ARE A HALO POPULATION
YOUNG and OPEN CLUSTERS ARE A DISC POPULATION

Portegies Zwart, McMillan & Gieles 2010

1. Relevant timescales of dense stellar systems:

MAIN PROPERTIES of COLLISIONAL stellar systems in the MILKY WAY

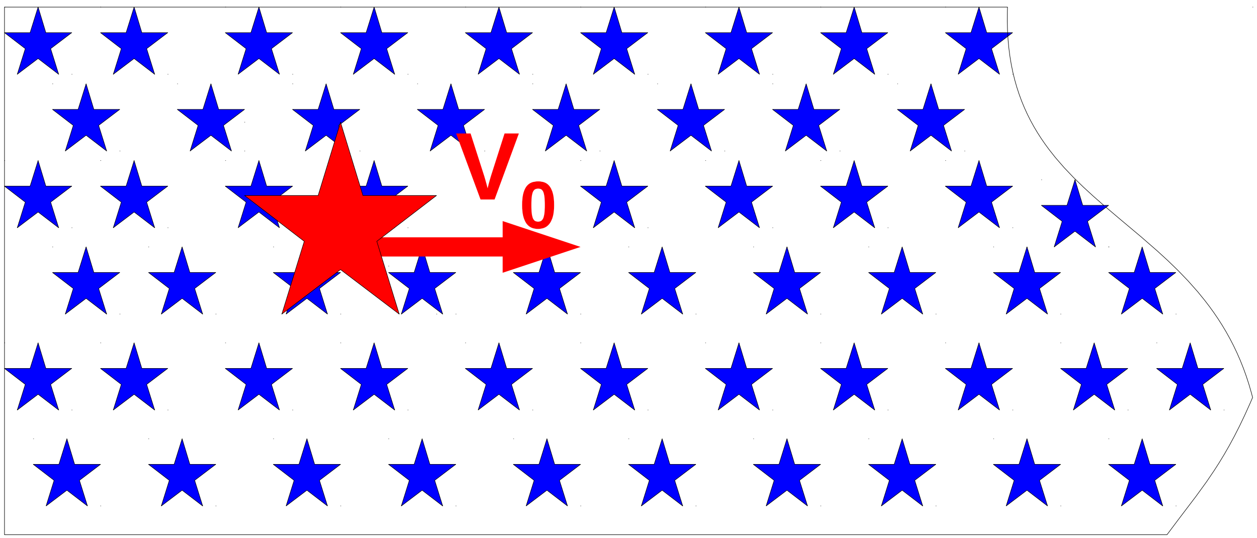


$$t_{\rm rlx} = 10\,{\rm Myr}\left(\frac{M_{\rm TOT}}{3500\,{\rm M}_\odot}\right)^{1/2}\left(\frac{r_{\rm hm}}{1\,{\rm pc}}\right)^{3/2}$$

Portegies Zwart, McMillan & Gieles 2010

DYNAMICAL FRICTION TIMESCALE

A body of mass *M*, traveling through an infinite & homogeneous sea of bodies (mass *m*) suffers a steady deceleration: the dynamical friction



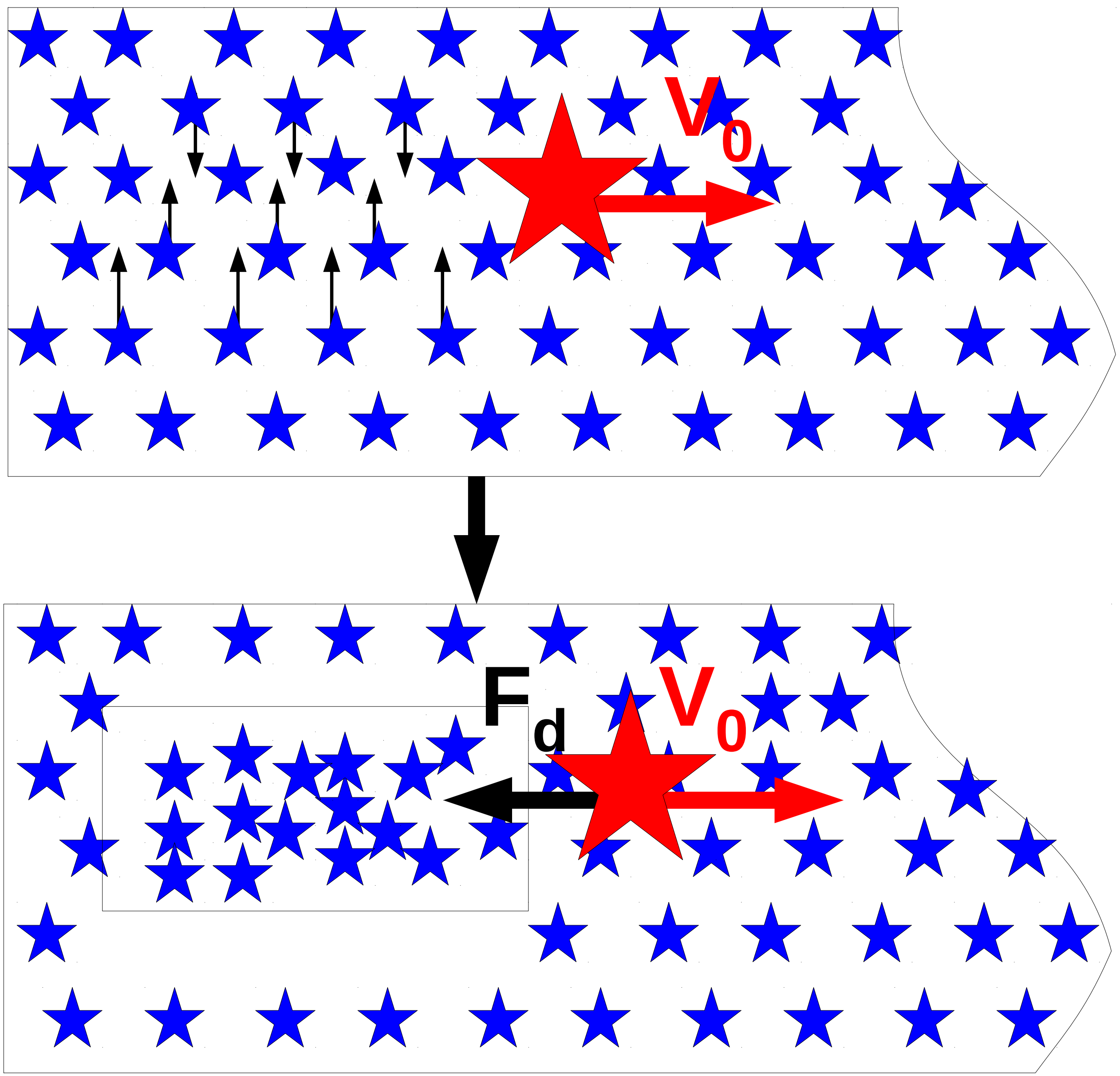
infinite & homogeneous sea: otherwise the body M would be deflected

The sea exerts a force **parallel and opposite** to the velocity V_0 of the body

It can be shown that DYNAMICAL FRICTION TIMESCALE is

$$t_{df} = \frac{3}{4\,(2\,\pi)^{1/2}\,G^2\,\ln\Lambda}\,\frac{\sigma^3(r)}{M\,\rho(r)}$$

BASIC IDEA of DYNAMICAL FRICTION:



The heavy body *M* attracts the lighter particles.

When lighter particles approach, the body *M* has already moved and leaves a local overdensity behind it.

The overdensity attracts the heavy body (with force F_d) and slows it down.

DYNAMICAL FRICTION vs 2-body RELAXATION:

what is the relation between two-body relaxation and dynamical friction?

DYNAMICAL FRICTION vs 2-body RELAXATION:

Dynamical friction timescale:

$$t_{df} = \frac{3}{4\,(2\,\pi)^{1/2}\,G^2\,\ln\Lambda}\,\frac{\sigma^3(r)}{M\,\rho(r)}$$

Two-body relaxation timescale:

$$t_{\rm rlx} = 0.34\,\frac{\sigma^3(r)}{G^2\,m_*\,\rho_*(r)\,\ln\Lambda}$$

They are relatives..

$$t_{df} \sim \frac{m}{M}\,t_{rlx}$$

SAME DRIVER: GRAVITATIONAL ENCOUNTERS

2. Early evolution of dense star systems

How do star clusters form? **BOH**

2. Early evolution of dense star systems

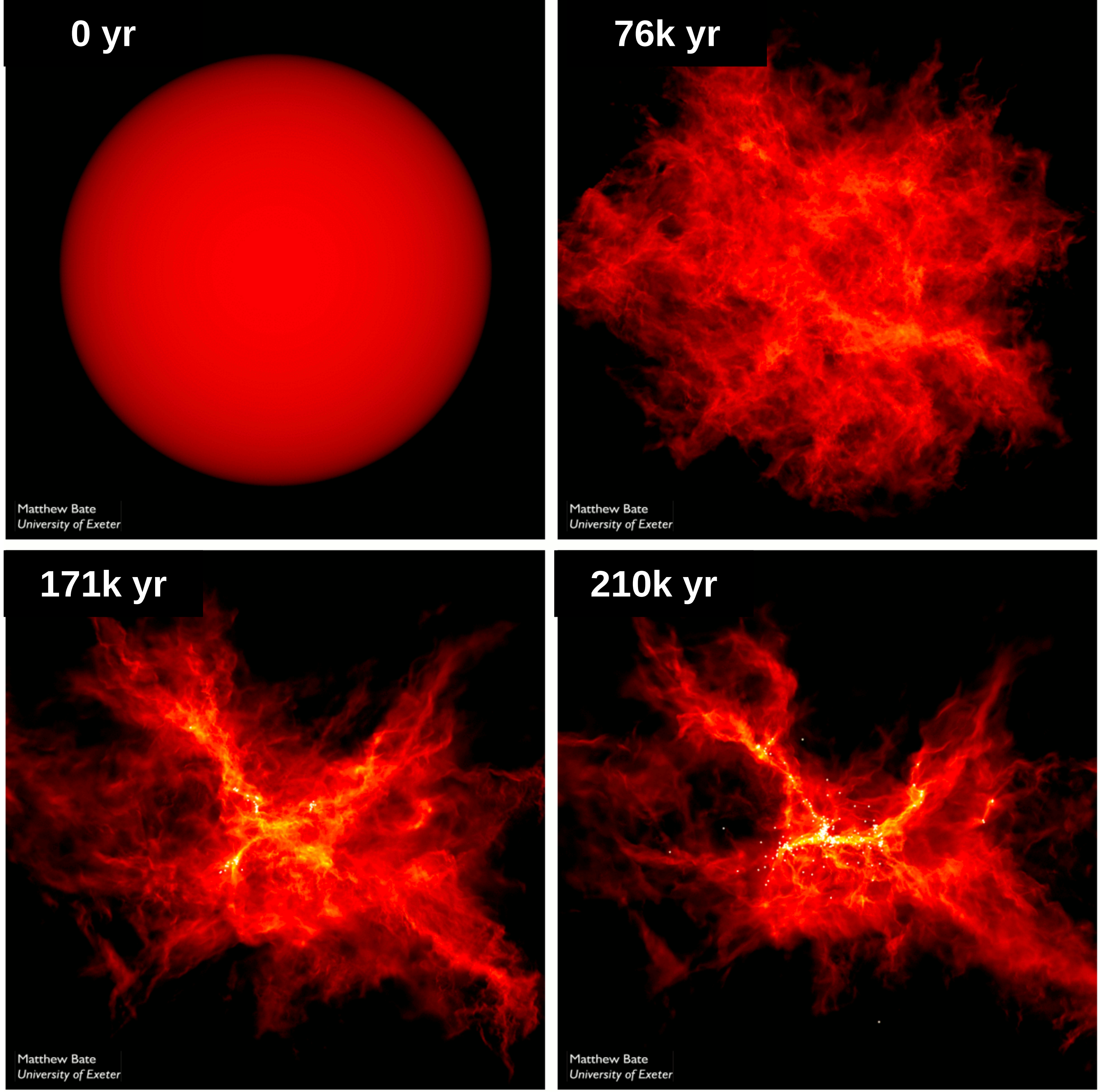
How do star clusters form?

* from giant molecular clouds

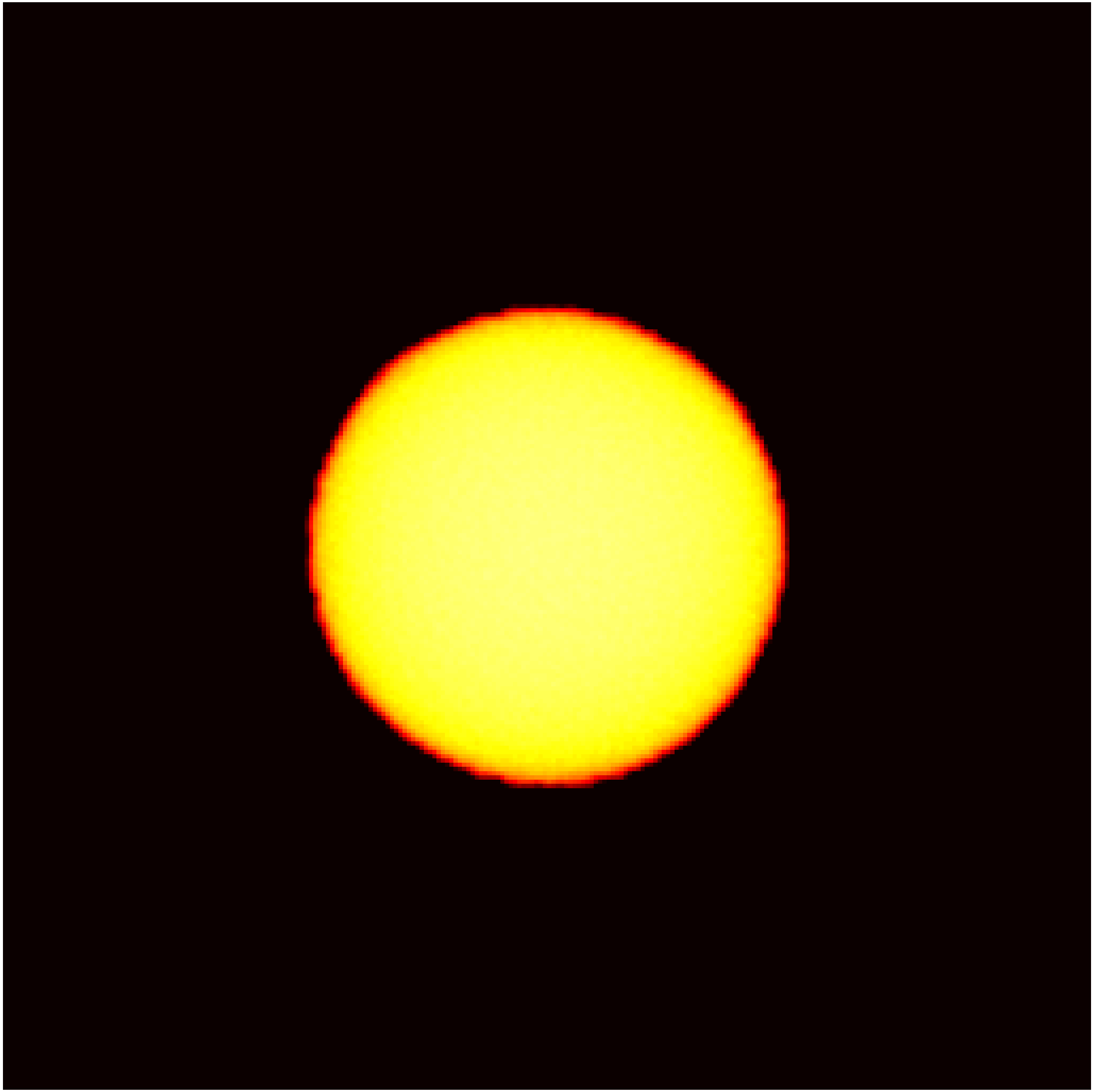
* possibly from aggregation of many sub-clumps
  (hierarchical formation)

CLOUD SIMULATION by Matthew Bate (Exeter):

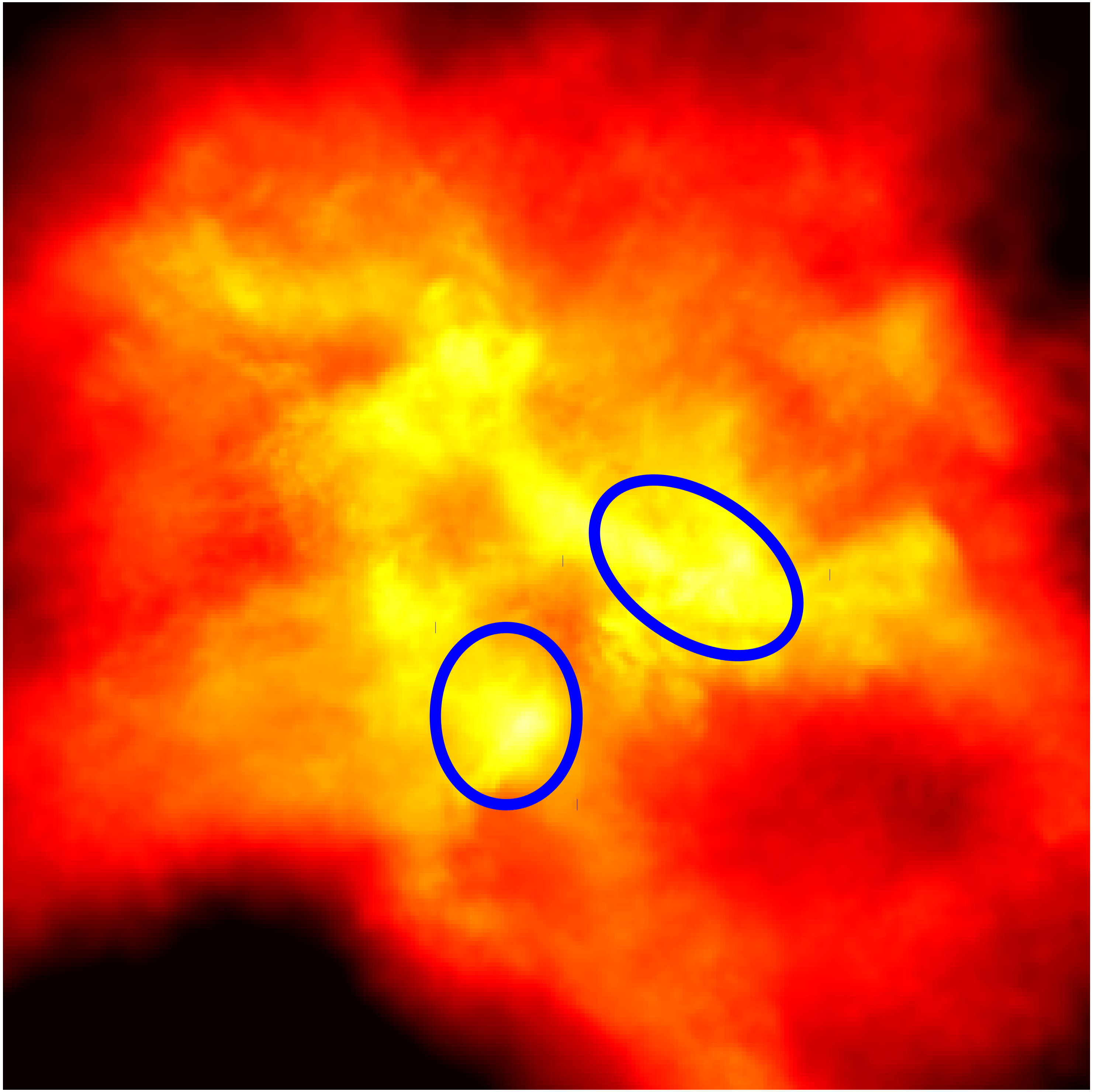
- gas
- gravity
- turbulence
- cooling /EOS
- fragmentation
- sink particles



CLOUD SIMULATION MOVIE



CLOUD SIMULATION MOVIE



SCs form from different cores of a molecular cloud

More filaments than spherical!

How do star clusters form?

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* can DIE by INFANT MORTALITY!!!

2. Early evolution of dense star systems:

INFANT MORTALITY:= clusters can die when GAS is removed

Embedded CLUSTERS

GAS REMOVAL →

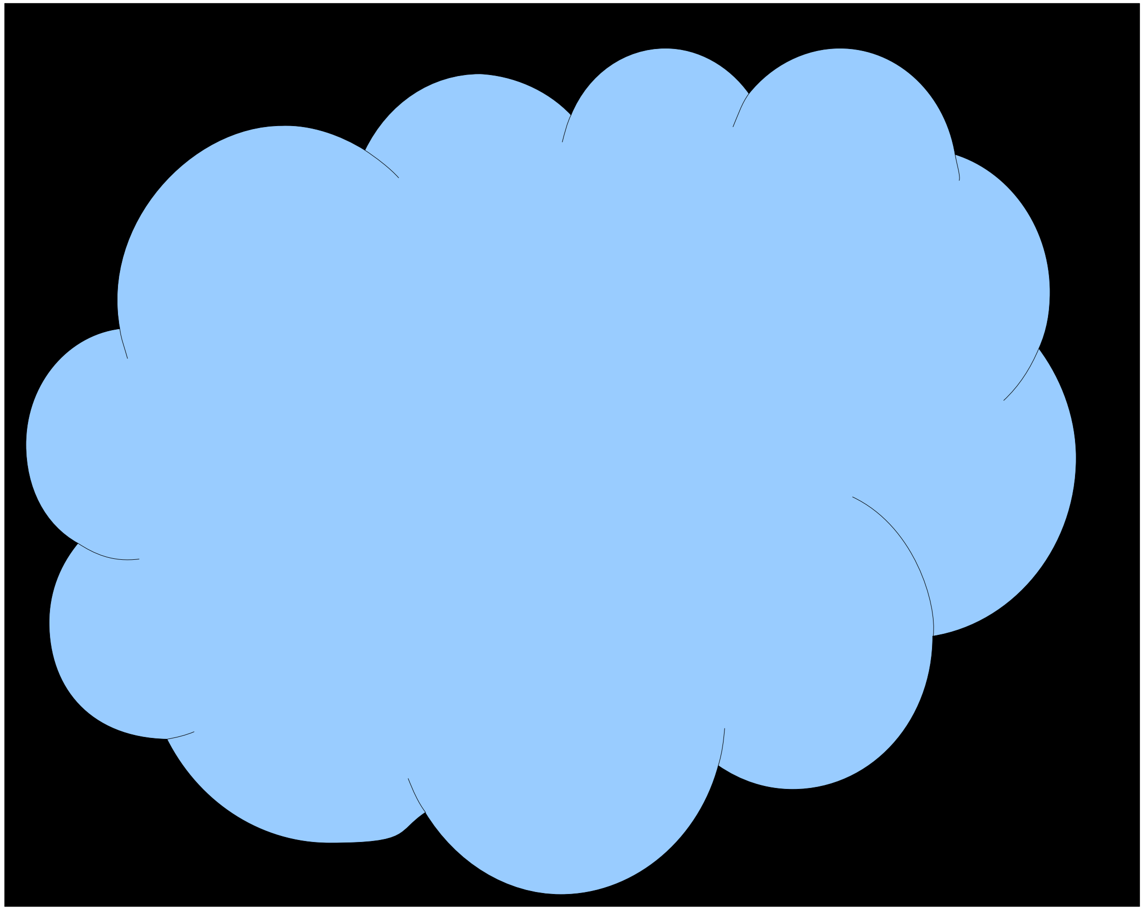
DENSE CLUSTERs
bound
$n \sim 10^{3-5}$ **pc**$^{-3}$ **(coll.)**
$\sim 10^{3-6}$ **stars**

OPEN CLUSTERS
loosely bound
$\sim 10^{3-4}$ **stars**

ASSOCIATIONS
unbound
$<10^{3}$ **stars**

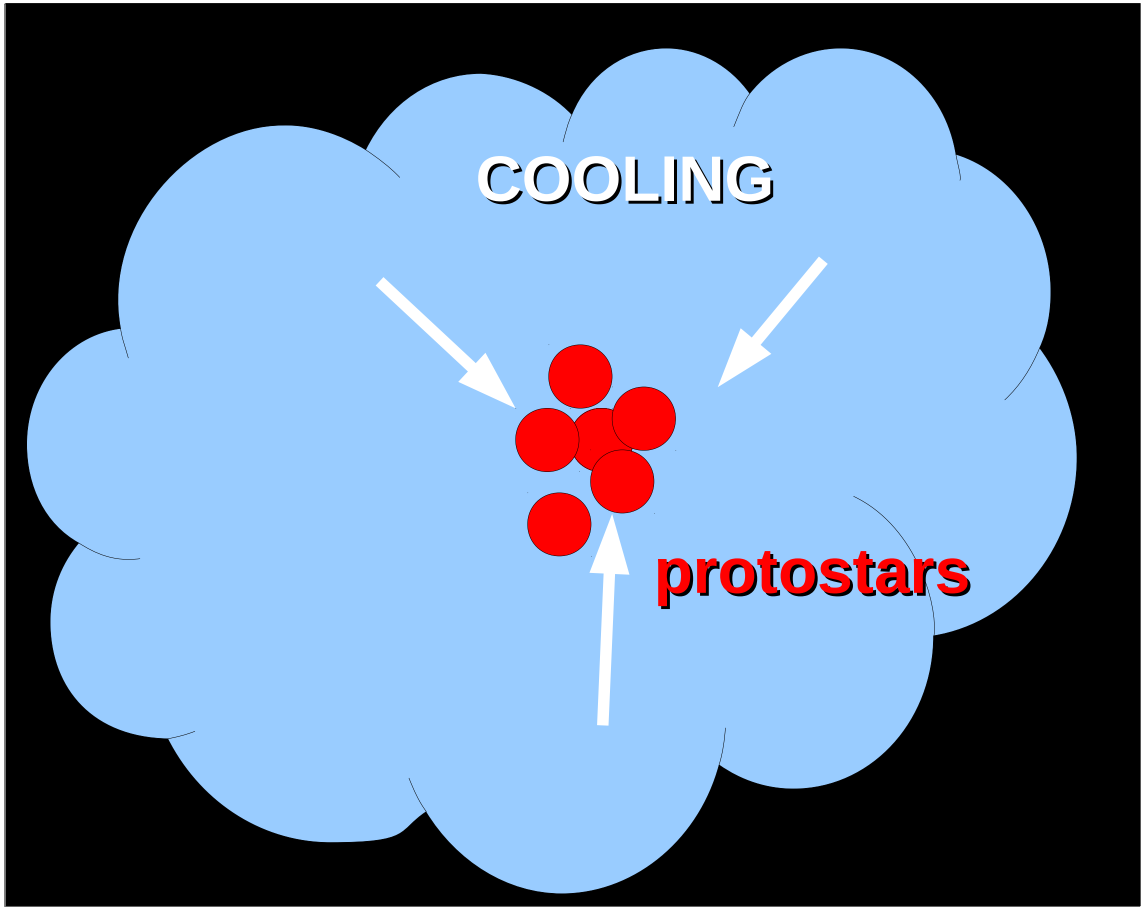
OPEN and DENSE STAR CLUSTERS as SURVIVORS of INFANT MORTALITY: how and with which properties?

FORMATION of STAR CLUSTERS: basic concepts



1 * giant molecular cloud: 10^{5-6} M⊙ of molecular gas, mainly H_2, in ~10 pc radius, at 10-100 K

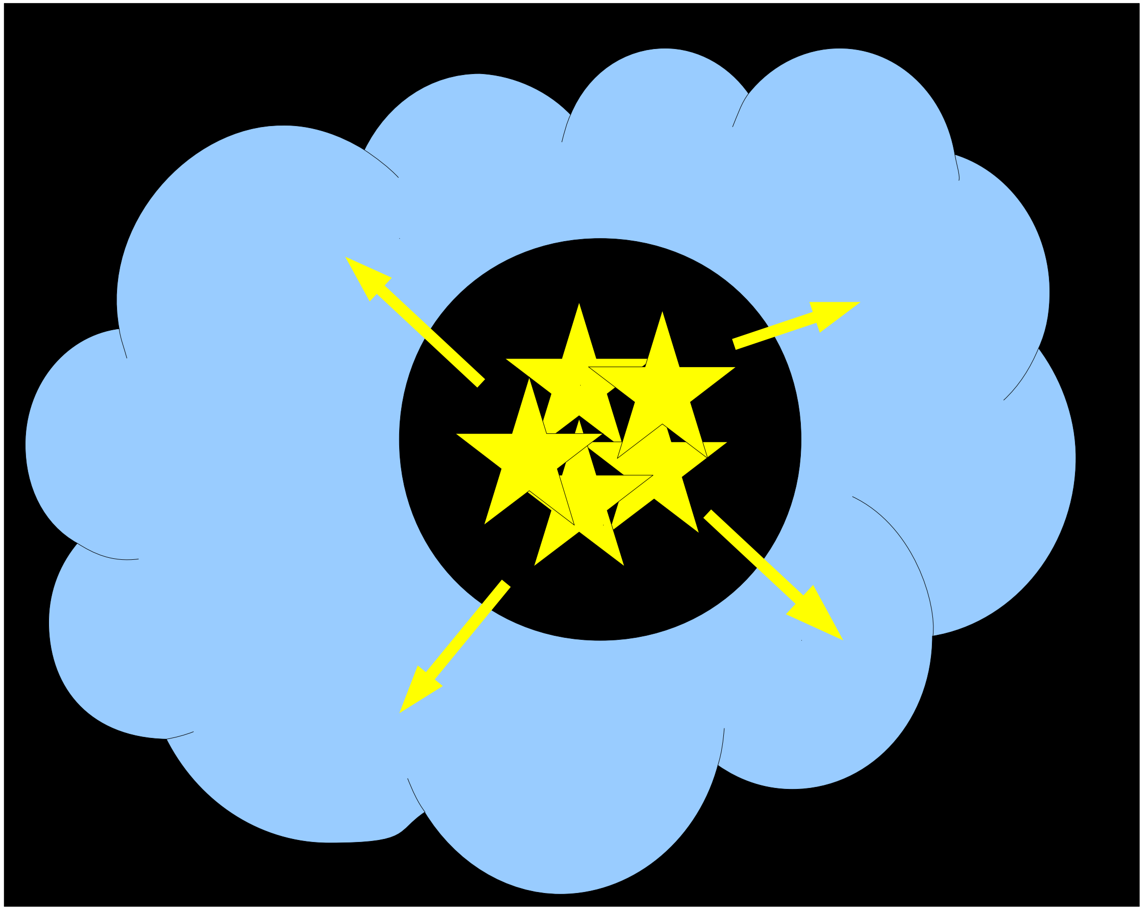
FORMATION of STAR CLUSTERS: basic concepts



1 * giant molecular cloud

2 * gas cools down and compresses → protostars form

FORMATION of STAR CLUSTERS: basic concepts



1 * giant molecular cloud

2 * gas cools down and compresses → protostars form

3 * protostars start irradiating and gas evaporates

FORMATION of STAR CLUSTERS: basic concepts

1 * giant molecular cloud

2 * gas cools down and compresses → protostars form

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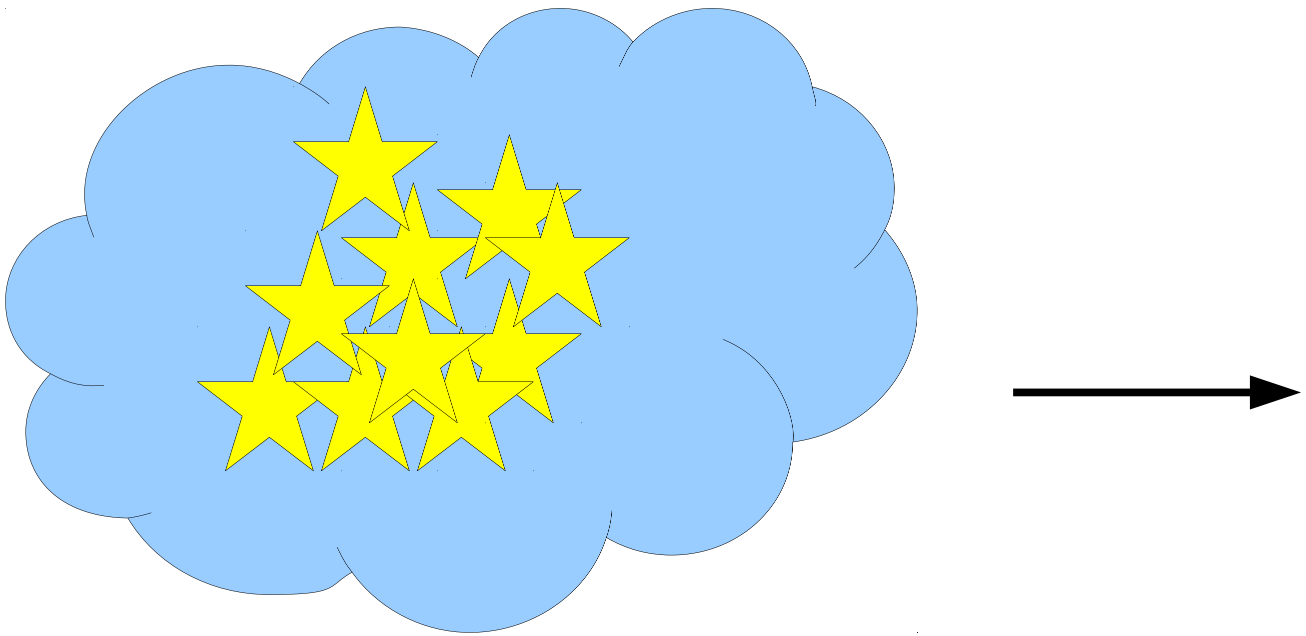
how many SCs survive gas evaporation?

2. Early evolution of dense star systems:

INFANT MORTALITY: a back to the envelope calculation

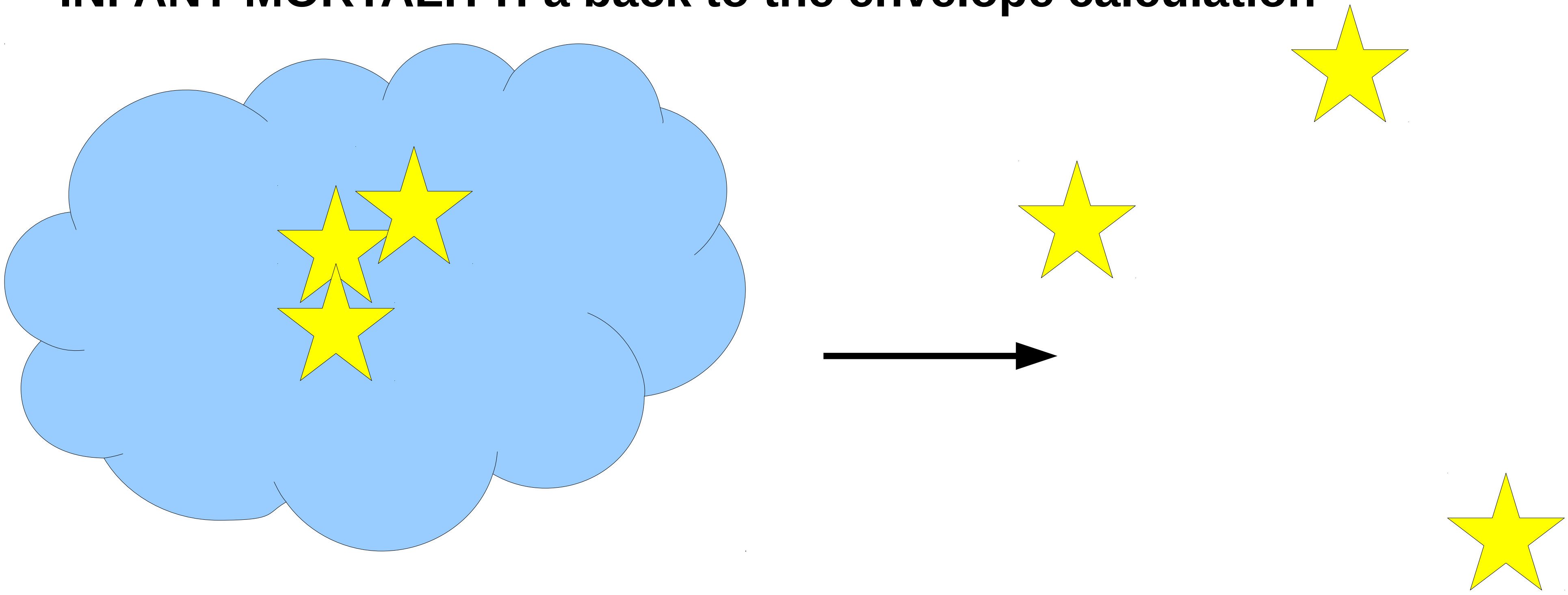


Intuitive argument:
$|W0|=G\ (Mgas+Mstar)^2/R$

If Mstar large with respect to Mgas, cluster remains bound

2. Early evolution of dense star systems:

INFANT MORTALITY: a back to the envelope calculation



Intuitive argument:
$|W0|=G\ (Mgas+Mstar)^2/R$

If Mstar large with respect to Mgas, cluster remains bound

If Mstar small with respect to Mgas, then cluster becomes unbound

INFANT MORTALITY: a back to the envelope calculation

(1) Velocity dispersion from virial theorem before gas removal:

$$\sigma_0^2 = \frac{G\,(M_{star} + M_{gas})}{R_0}$$

(2) Energy after gas removal (hypothesis of instantaneous gas removal):

$$E = \frac{1}{2}\,M_{star}\,\sigma_0^2 - \frac{G\,M_{star}^2}{R_0}$$

(3) Energy after new virialization:

$$E = -\frac{G\,M_{star}^2}{2\,R}$$

New cluster size:
- From (2) = (3)

$$-\frac{G\,M_{star}^2}{2\,R} = \frac{1}{2}\,M_{star}\,\sigma_0^2 - \frac{G\,M_{star}^2}{R_0}$$

Hills 1980, ApJ, 225, 986

INFANT MORTALITY: a back to the envelope calculation

New cluster size:
- Using (1)

$$-\frac{M_{star}}{2\,R} = \frac{1}{2}\,\frac{(M_{star}+M_{gas})}{R_0} - \frac{M_{star}}{R_0}$$

-Rearranging

$$R = R_0\,\frac{M_{star}}{M_{star}+M_{gas}}\,\frac{1}{\left(2\,\frac{M_{star}}{M_{star}+M_{gas}}-1\right)}$$

R>0 only if

$$\boxed{\frac{M_{star}}{M_{star}+M_{gas}} > 0.5}$$

Hills 1980, ApJ, 225, 986

INFANT MORTALITY: the complete picture

-DEPENDENCE on SFE : <30% disruption

-DEPENDENCE on t_gas:

explosive removal: t_gas<< t_cross
[smaller systems]

adiabatic removal: t_gas >~ t_cross
[dense clusters]

-DEPENDENCE on the (+/-) VIRIAL state of the embedded cluster

-DEPENDENCE on Z: metal poor clusters more compact than metal rich

Hills 1980; Lada & Lada 2003; Bastian & Goodwin 2006; Baumgardt & Kroupa 2007; Bastian 2011; Pelupessy & Portegies Zwart 2011

How do star clusters form?

* from giant molecular clouds

* possibly from aggregation of many sub-clumps
  (hierarchical formation)

* can DIE by INFANT MORTALITY!!!

* if they survive infant mortality, gas-less star clusters can be
  described with distribution functions:
  PLUMMER SPHERE
  ISOTHERMAL SPHERE
  LOWERED ISOTHERMAL SPHERE
  KING MODEL

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DON'T YOU NOTICE ANYTHING STRANGE IN THIS SLIDE??

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3. Equilibrium models

DISTRIBUTION FUNCTION or PHASE SPACE DENSITY

$$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v}$$

Number of stars in the infinitesimal volume $d^3\mathbf{x}$ and in the small range of velocities d^3v

DISTRIBUTION FUNCTIONS ARE WELL DEFINED ONLY FOR COLLISIONLESS SYSTEMS!!

Because they can be CONTINUOUS only if potential is smooth

BUT if system is COLLISIONAL potential is not smooth, particles jump from one side to the other of the phase space!

For a short time even a collisional system can be defined by a distribution function (not correct but useful in practice)

Then 2-body relaxation produces jumps and collisional system passes from one equilibrium to another

DISTRIBUTION FUNCTION or PHASE SPACE DENSITY

* Equations of motion in the phase space using distribution functions can be expressed with **collisionless Boltzmann equation (CBE)**

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1,6} \frac{\partial x_i}{\partial t} \frac{\partial f}{\partial x_i} = 0$$

same as continuity equation for fluids: valid only if no jumps

* JEANS theorem: any steady-state solution of the CBE is a function of the integrals of motion and any function of the integrals of motion is a steady-state solution of the CBE

* Poisson Vlasov equation describes relation between gravity force and its Sources (same as Gauss)

$$\nabla^2 \phi = 4\,\pi\,G\,\rho$$

* Often potentials and energies are given as

RELATIVE potential:

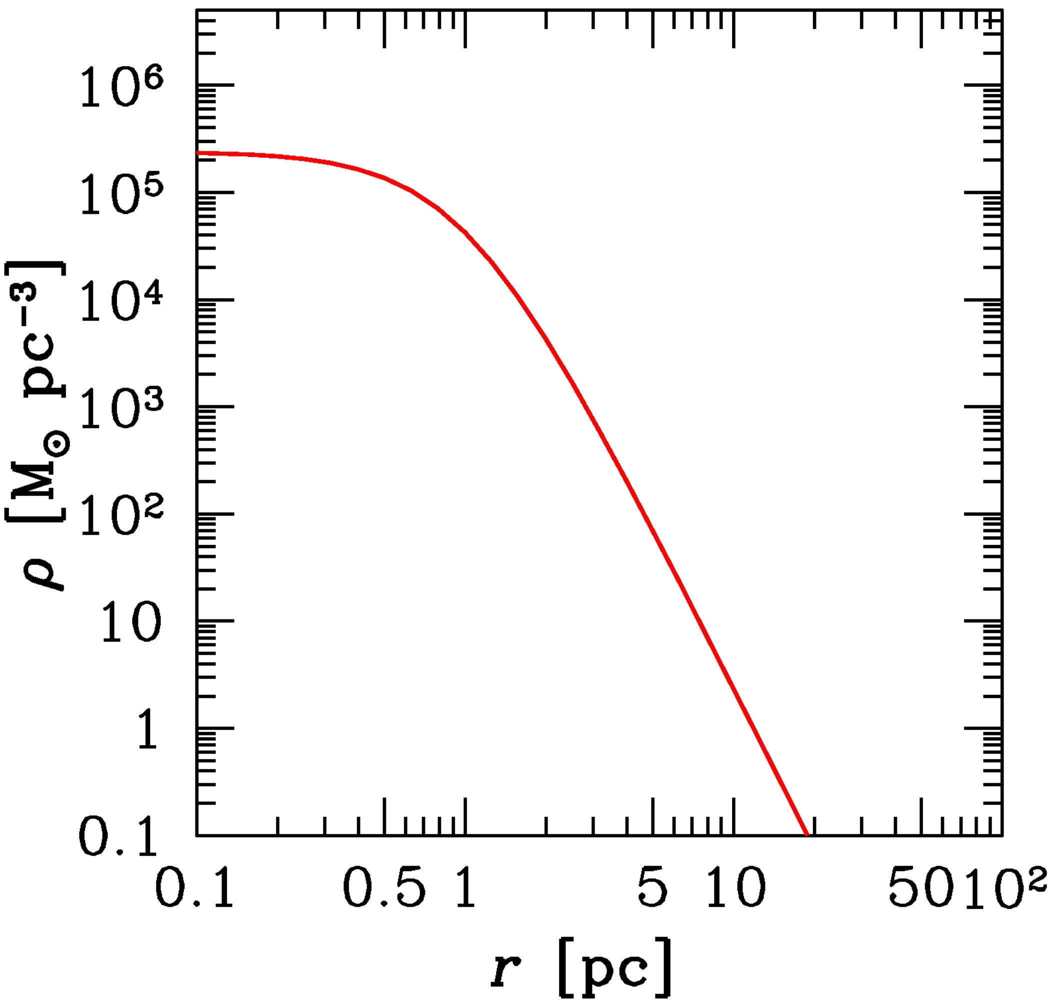
$$\Psi = -\Phi + \Phi_0$$

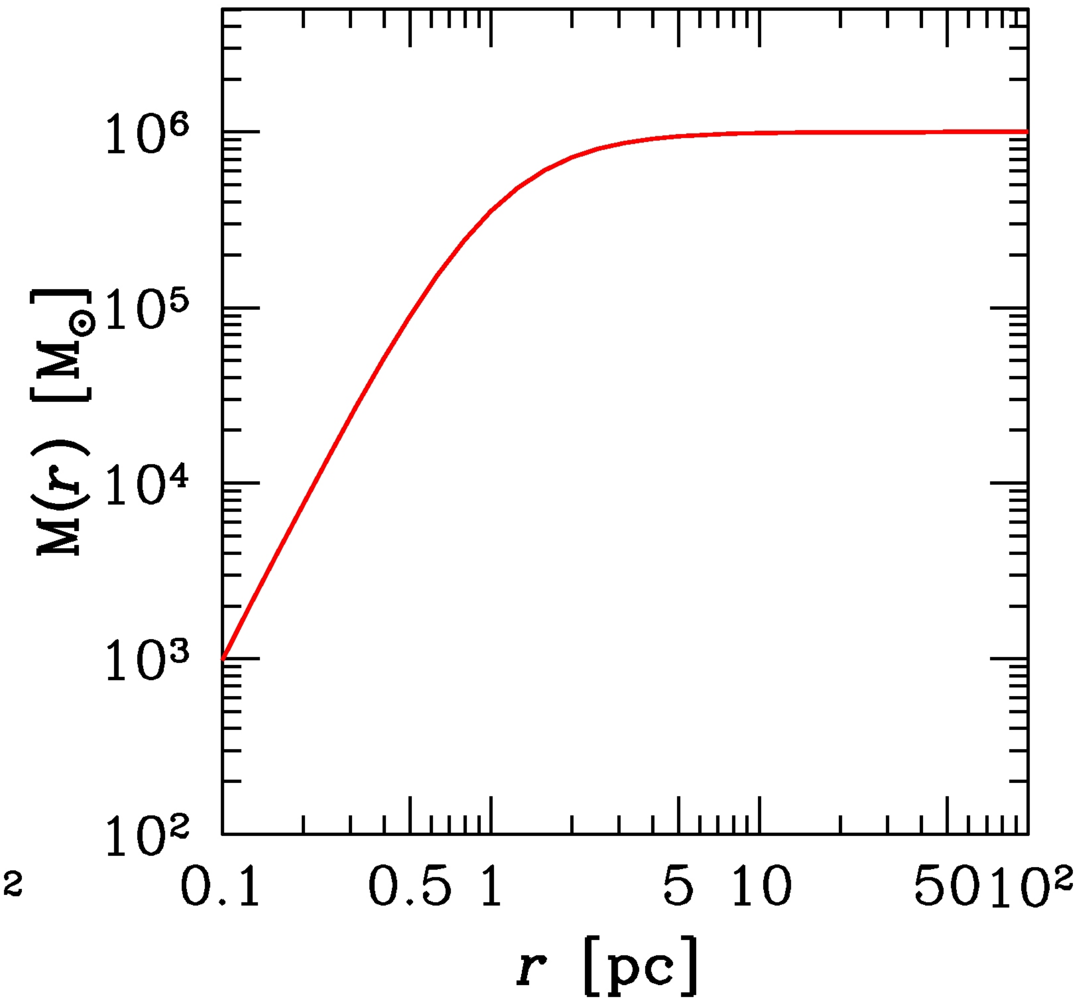
RELATIVE energy:

$$\varepsilon = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$$

3. Equilibrium models:

Plummer sphere





Plummer sphere

Isotropic velocity distribution function: $f(E) \propto \begin{cases} (-E)^p & if\, E < 0 \\ 0 & if\, E \geq 0 \end{cases}$

if *p*=1 corresponds to potential $\phi(r) = -\dfrac{G\,M}{\left(r^2 + a^2\right)^{1/2}}$

From Poisson equation $\nabla^2\phi = 4\,\pi\,G\,\rho$

We derive density

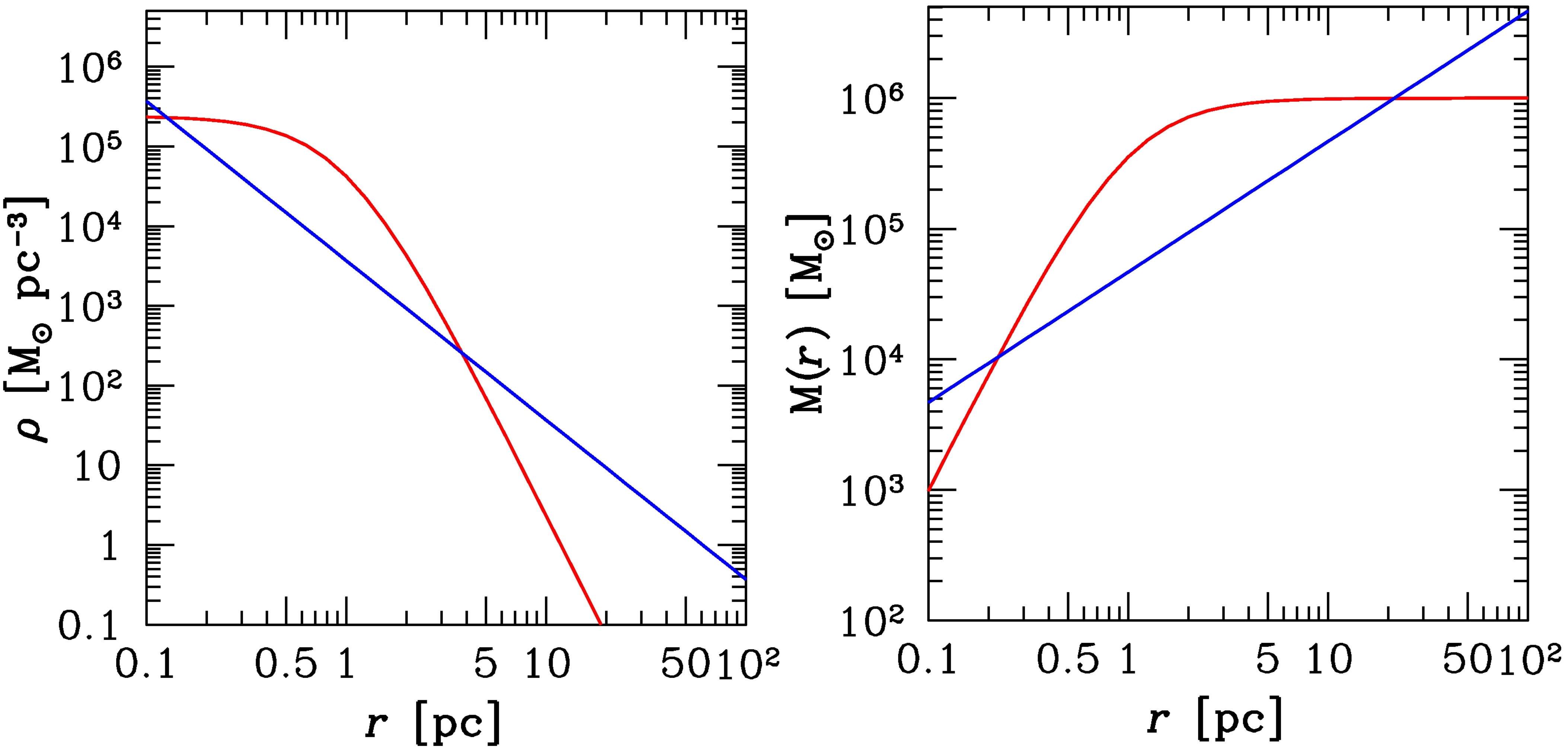
$$\rho(r) = \frac{M}{\frac{4}{3}\,\pi a^3}\,\frac{1}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{5/2}}$$

and corresponding mass

$$M(r) = \frac{M}{a^3}\,\frac{r^3}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{3/2}}$$

3. Equilibrium models:

Isothermal sphere



3. Equilibrium models:

Isothermal sphere

* Why isothermal? From formalism of ideal gas $P = \frac{\kappa_B}{\mu m_p} \rho T$

If T= const → P = const x ρ

* For politropic equation of state $P = \kappa \rho^{\gamma}$
is isothermal if γ = 1

γ = 1

if we assume
hydrostatic equilibrium

$$\frac{d\phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr} = -\frac{\kappa}{\rho}\frac{d\rho}{dr}$$

we derive the potential

$$\phi = -\kappa \ln\left(\frac{\rho}{\rho_c}\right)$$

using Poisson's equation we find

$$\rho(r) = \frac{\sigma^2}{2\,\pi\,G} r^{-2}$$

expressing the constant *k* with some physical quantities

PROBLEMS of isothermal sphere

1) DENSITY goes to infinity if radius goes to zero

$$\rho(r) = \frac{\sigma^2}{2\,\pi\,G} r^{-2}$$

2) MASS goes to infinity if radius goes to infinity

$$M(r) = 4\,\pi \int_0^r \rho(r)\, r^2\, dr = \frac{2\,\sigma^2}{G} r$$

3. Equilibrium models:

Non-singular isothermal sphere or King model

1) King model (also said non-singular isothermal sphere) solves the problem at centre by introducing a CORE

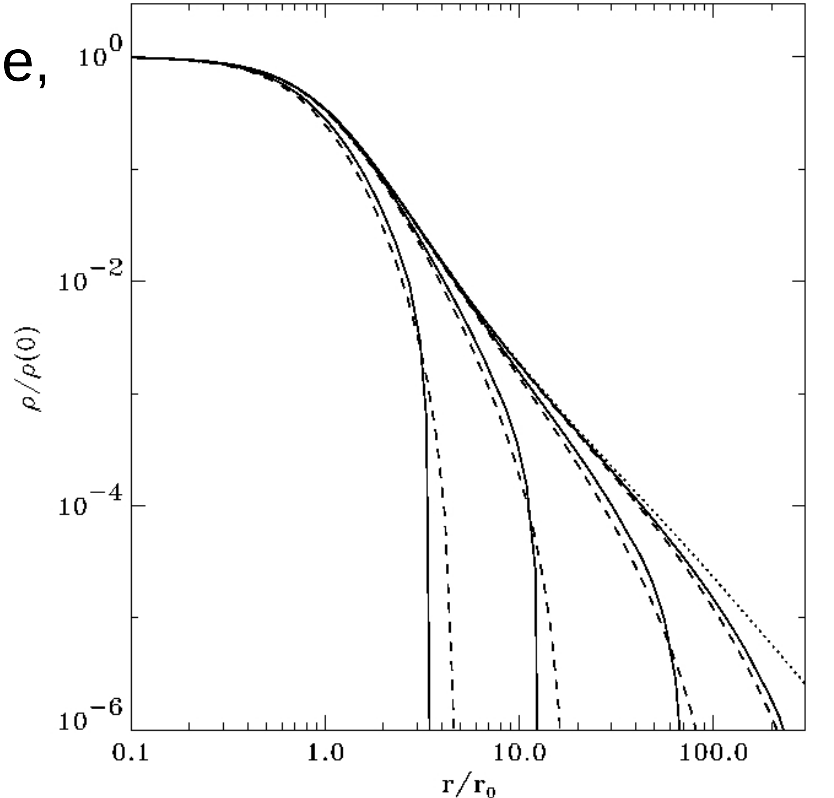
$$\tilde{\rho} = \frac{\rho}{\rho_0} \qquad \tilde{r} = \frac{r}{r_0} \qquad r_0 = \sqrt{\frac{9\,\sigma^2}{4\,\pi\,G\,\rho_0}}$$

r_0 is the radius at which the projected density falls to ~half

with the core, ρ has a difficult analytical shape, but can be approximated
with the singular isothermal sphere for $r \gg r_0$
and with

$$\rho(r) = \rho_0 \frac{1}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{3/2}}$$

for $r \lesssim 2\, r_0$



Lowered non-singular isothermal sphere or lowered King model

2) LOWERED non-singular isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

VELOCITY DISTRIBUTION FUNCTION:

$$f(E) \propto \begin{cases} \kappa\left(e^{-B\,E} - e^{-B\,E_e}\right) & if\,E < E_e \\ 0 & if\,E \geq E_e \end{cases}$$

DENSITY EXPRESSION:

$$\rho_K(\Psi) = \rho_1 \left[\exp\left(\Psi/\sigma^2\right)\mathrm{erf}\left(\frac{\sqrt{\Psi}}{\sigma}\right) - \sqrt{\frac{4\,\Psi}{\pi\,\sigma^2}}\left(1 + \frac{2\,\Psi}{3\,\sigma^2}\right)\right]$$

Relative potential (Ψ)

Error function (erf)

Lowered non-singular isothermal sphere or lowered King model

2) LOWERED non-singular isothermal sphere := all King models where mass is truncated at a certain radius (does not go to infinity)

TIDAL RADIUS (r_t):

Radius at which Ψ = 0 (and ρ=0)

→ we can define

$$0 = \Psi(r_t) = -\Phi(r_t) + Const$$

$$Const = \Phi(r_t) = -\frac{G\,M(r_t)}{r_t}$$

$$\Phi(0) = \Phi(r_t) - \Psi(0)$$

$$W_0 = \Psi(0)/\sigma^2 \qquad c = \log_{10}(r_t/r_0)$$

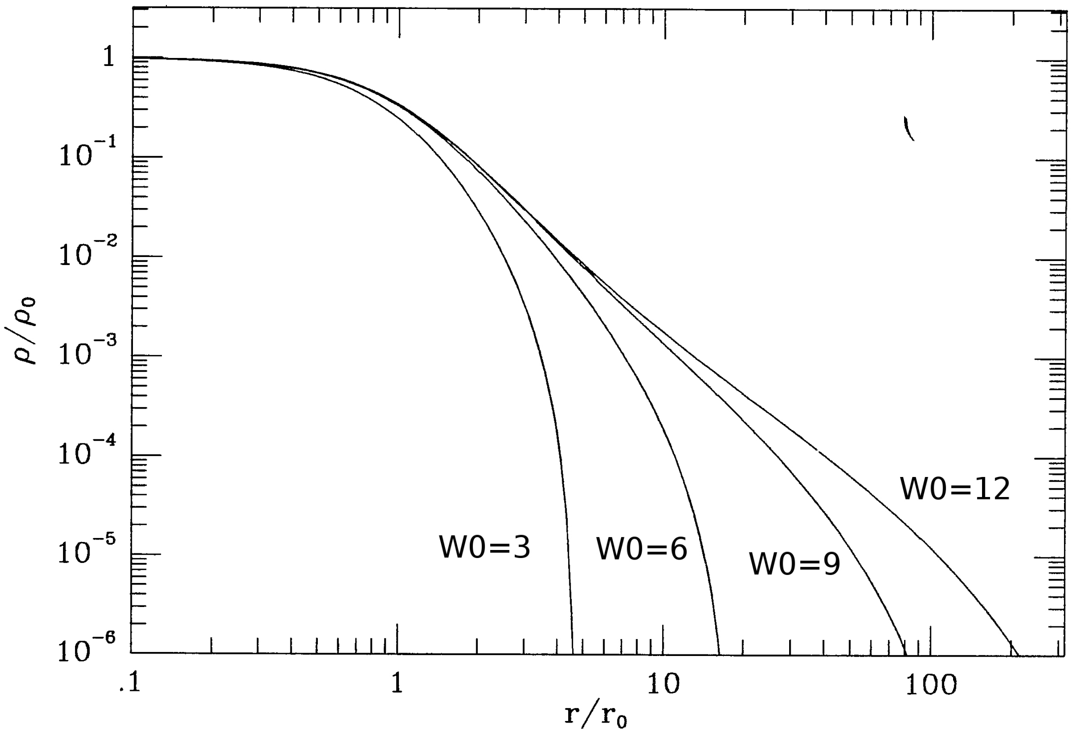
W_0 ← Dimensionless central potential

$\log$ ← concentration

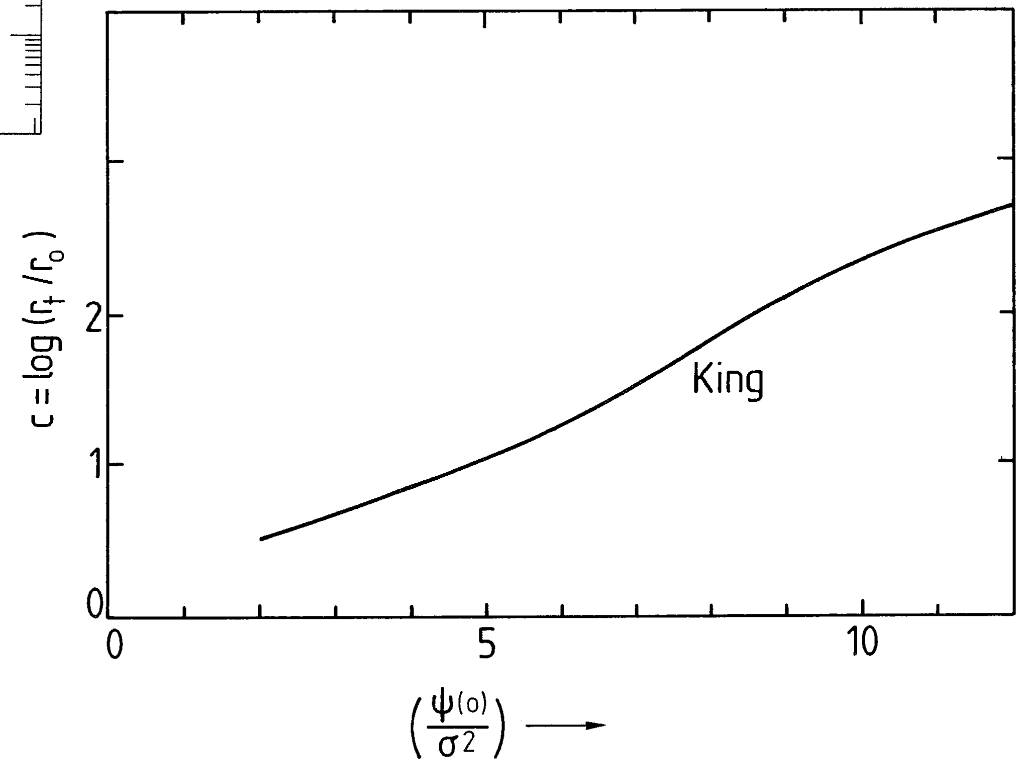
Most important parameters of the King model.

3. Equilibrium models:

Lowered non-singular isothermal sphere or lowered King model



ρ/ρ0 versus r/r0

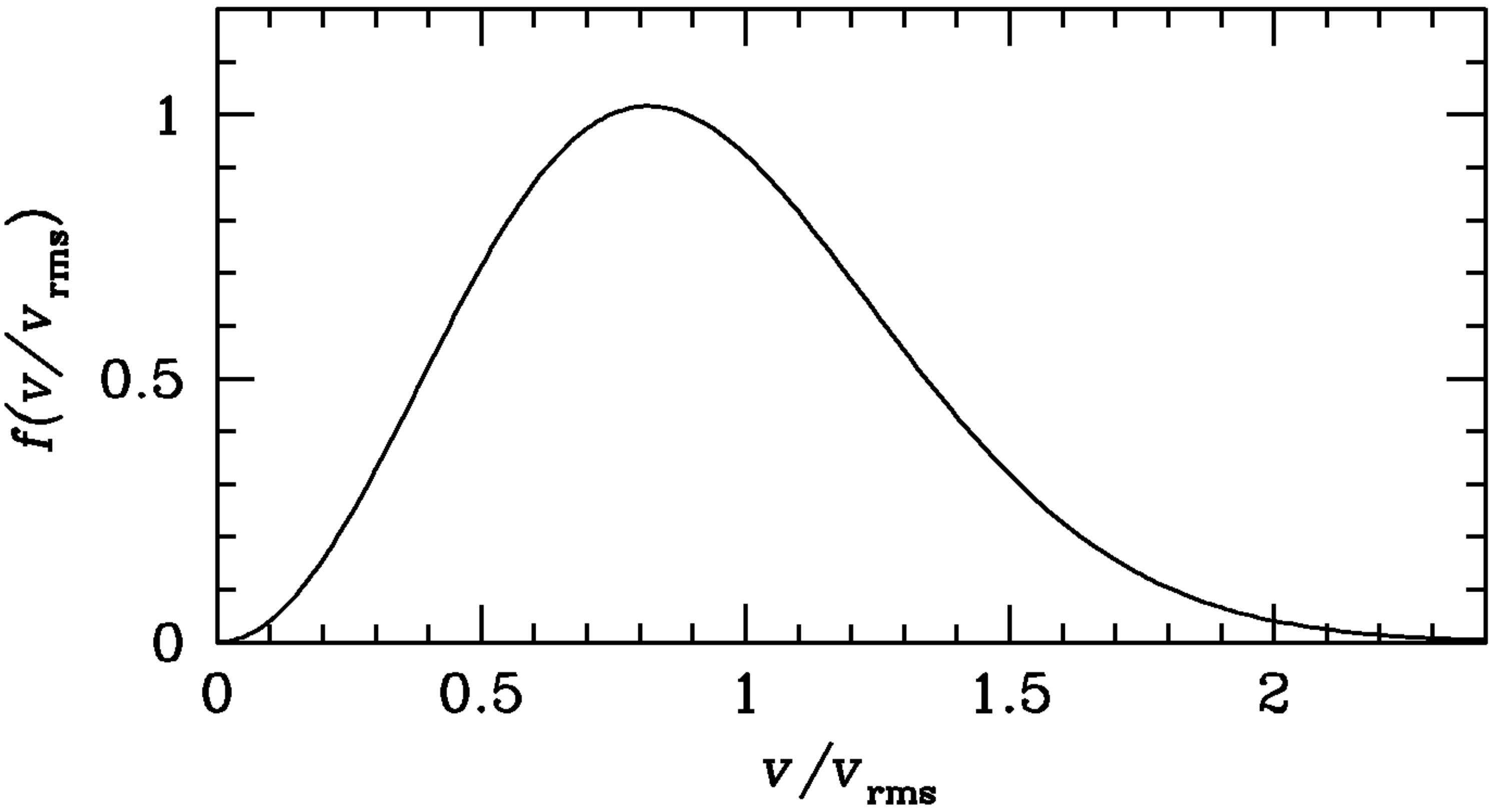


***c* versus W0**

Binney & Tremaine 1987

Lowered non-singular isothermal sphere or lowered King model

NOTE: VELOCITY DISTRIBUTION FUNCTION is the MAXWELLIAN for isothermal sphere and truncated Maxwellian for lowered non-singular isothermal sphere!!!



$$f(v) \propto v^2\, e^{-m\, v^2/(2\, k_b\, T)}$$

4. N-body simulations of star clusters + star evolution

How do star clusters form?

* from giant molecular clouds

* possibly from aggregation of many sub-clumps (hierarchical formation)

* can DIE by INFANT MORTALITY!!!

* if they survive infant mortality, gas-less star clusters can be described with distribution functions:
  - PLUMMER SPHERE
  - ISOTHERMAL SPHERE
  - LOWERED ISOTHERMAL SPHERE
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* collisional dynamics of a star cluster must be studied with dedicated N-body simulations that resolve two-body encounters!!!

4. N-body simulations of star clusters + star evolution

*** Describing a collisional system with N-body simulations requires to resolve two-body encounters properly**

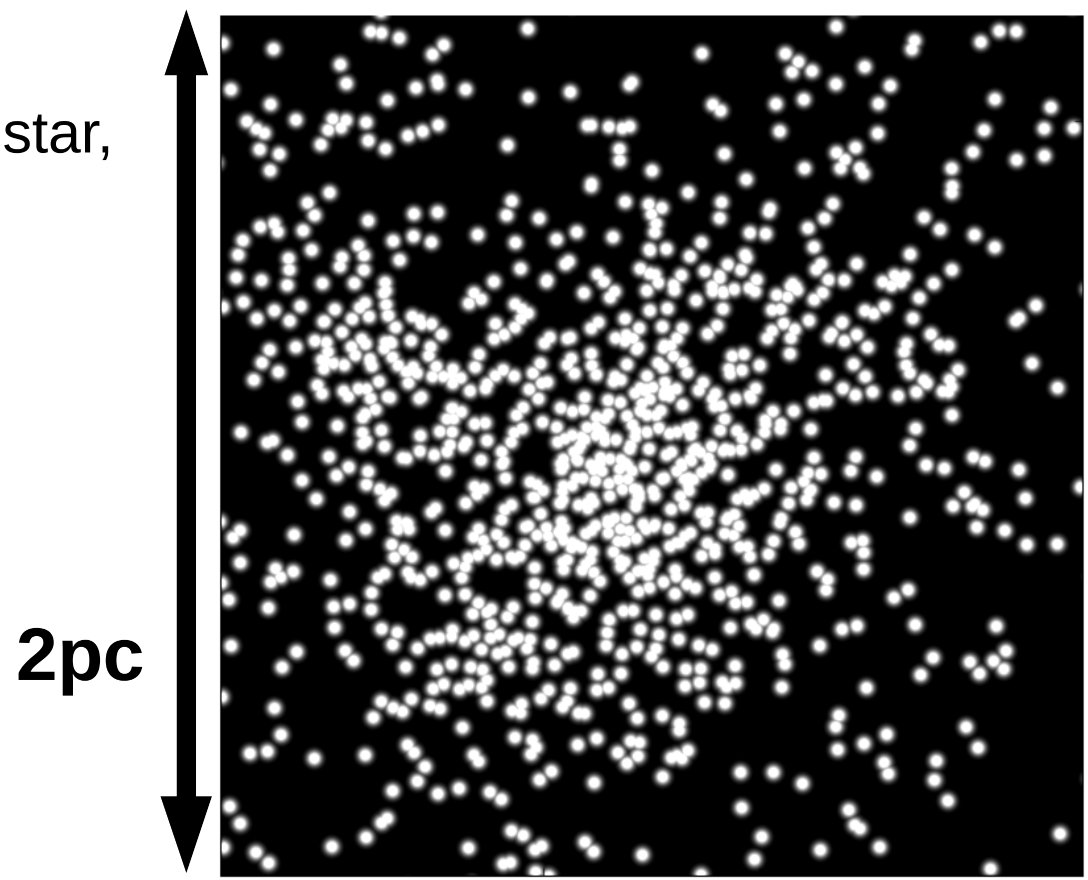
→ solve Newton's equation directly, no approximations

$$\ddot{\vec{r}}_i = -G \sum_{j \neq i} m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

→ **DIRECT N-body codes** scaling as N^2

* Each particle ~ corresponds to a star, with its mass and radius

* Suitable for running on graphics processing units (GPUs)

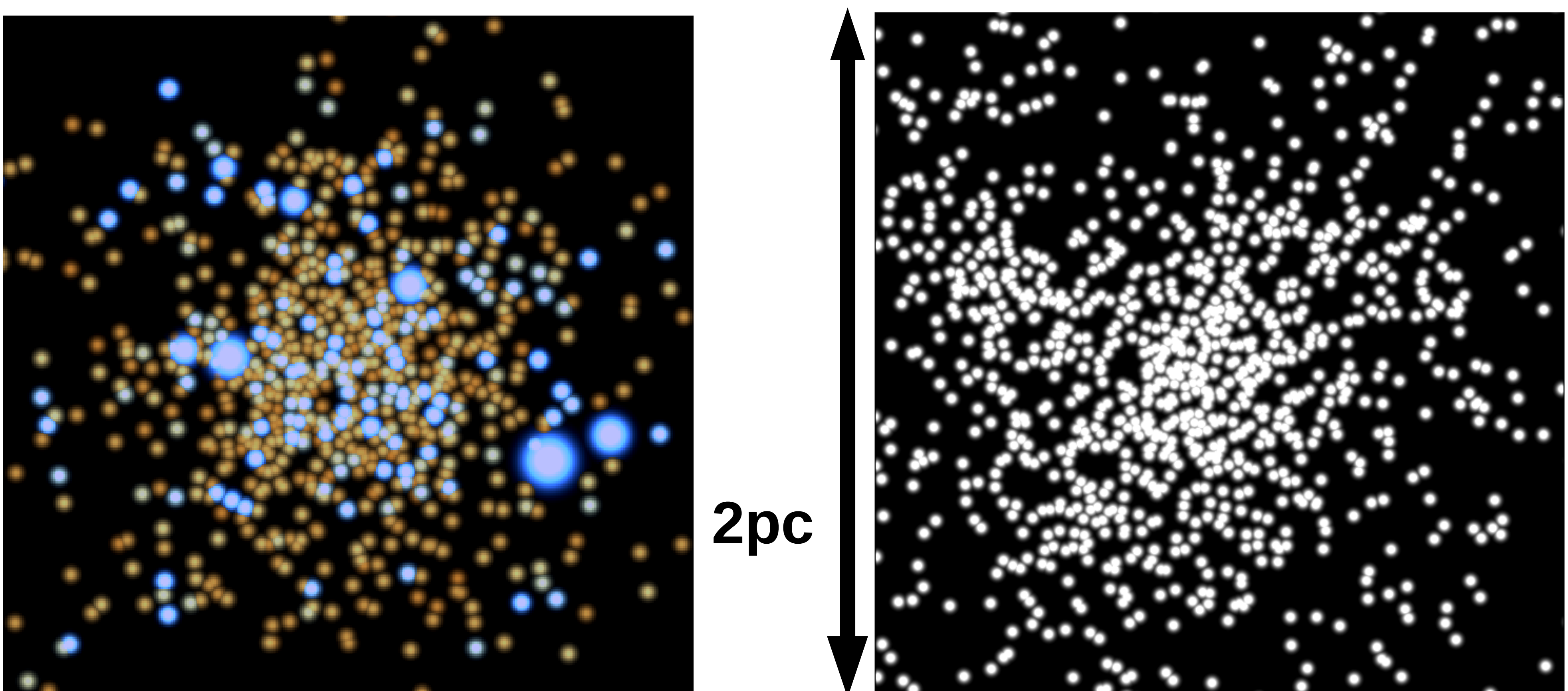


4. N-body simulations of star clusters + star evolution

* DIRECT N-body codes can be couplet with star evolution through a POPULATION SYNTHESIS approach

→ each particle DOES correspond to a single star with its own mass, radius, metallicity, luminosity, temperature THAT CHANGE IN TIME ACCORDING TO star evolution

Gravity + stellar evolution | Gravity only



4. N-body simulations of star clusters + star evolution

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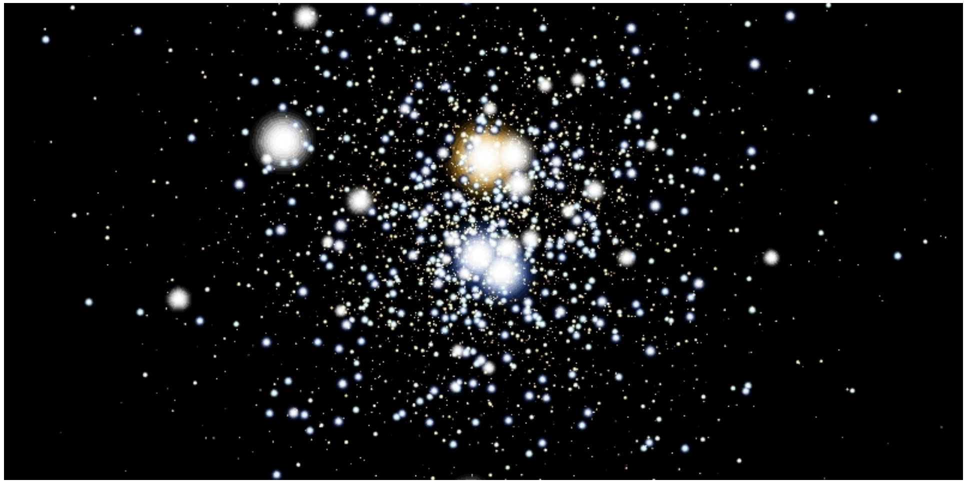
Credits: A. Geller, Northwestern
Movie 5
Movie 6



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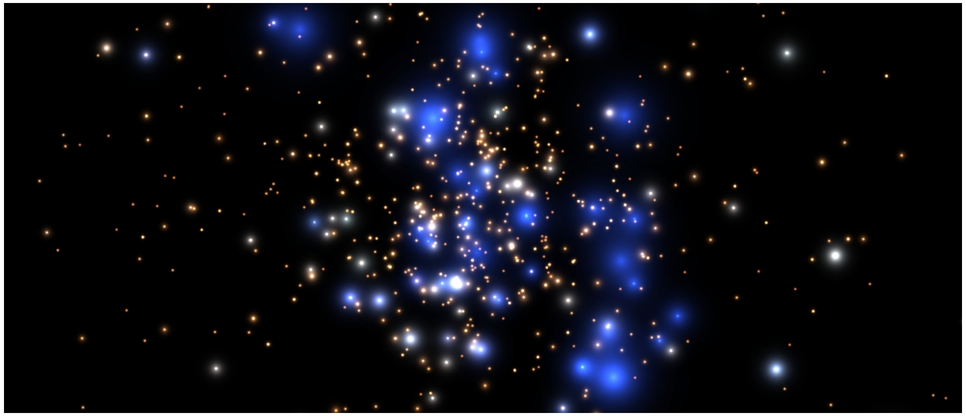

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