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LECTURES on COLLISIONAL DYNAMICS:

3. BINARIES and 3-BODY ENCOUNTERS

BINARIES as ENERGY RESERVOIR

Binaries have a energy reservoir (their internal energy) that can be exchanged with stars.

INTERNAL ENERGY: total energy of the binary – kinetic energy of the centre-of-mass

$$E_{int} = \frac{1}{2}\,\mu\,v^2 - \frac{G\,m_1\,m_2}{r}$$

where m_1 and m_2 are the mass of the primary and secondary member of the binary, μ is the reduced mass (:= $m_1\,m_2/(m_1+m_2)$).
r and v are the relative separation and velocity.

$E_{int}<0$ if the binary is bound

Note that E_{int} can be interpreted as the energy of the 'reduced particle': a fictitious particle of mass μ orbiting in the potential $-\,G\,m_1\,m_2/r$

BINARIES as ENERGY RESERVOIR

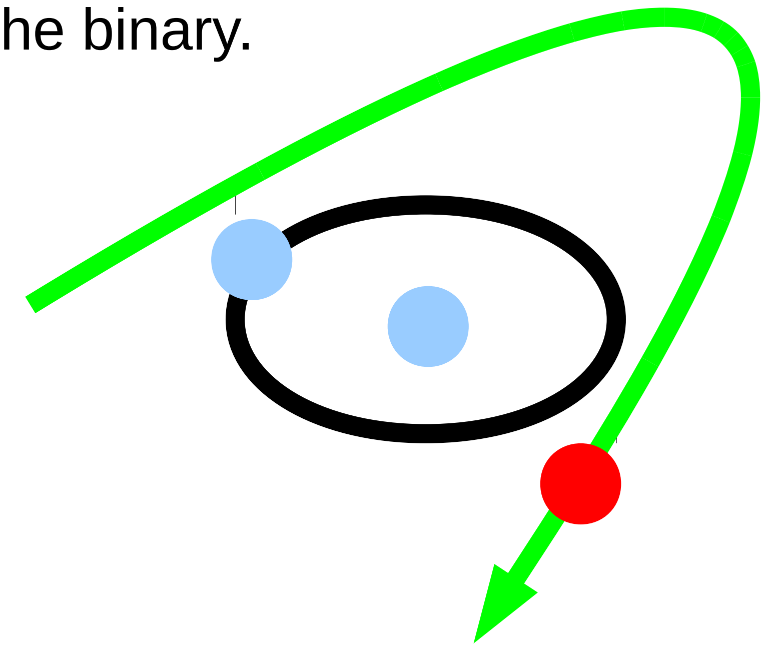
As far as the binary is bound, the orbit of the reduced particle is a Kepler ellipse with semi-major axis *a*. Thus, the energy integral of motion is

$$E_{int} = -\frac{G\, m_1\, m_2}{2\, a} = -E_b$$

where E_b is the **BINDING ENERGY** of the binary.

THE ENERGY RESERVOIR of BINARIES can be EXCHANGED with stars:

during a **3-BODY INTERACTION**, i.e. an interaction between a binary and a single star, the single star can either EXTRACT INTERNAL ENERGY from the binary or lose a fraction of its kinetic energy, which is converted into internal energy of the binary.



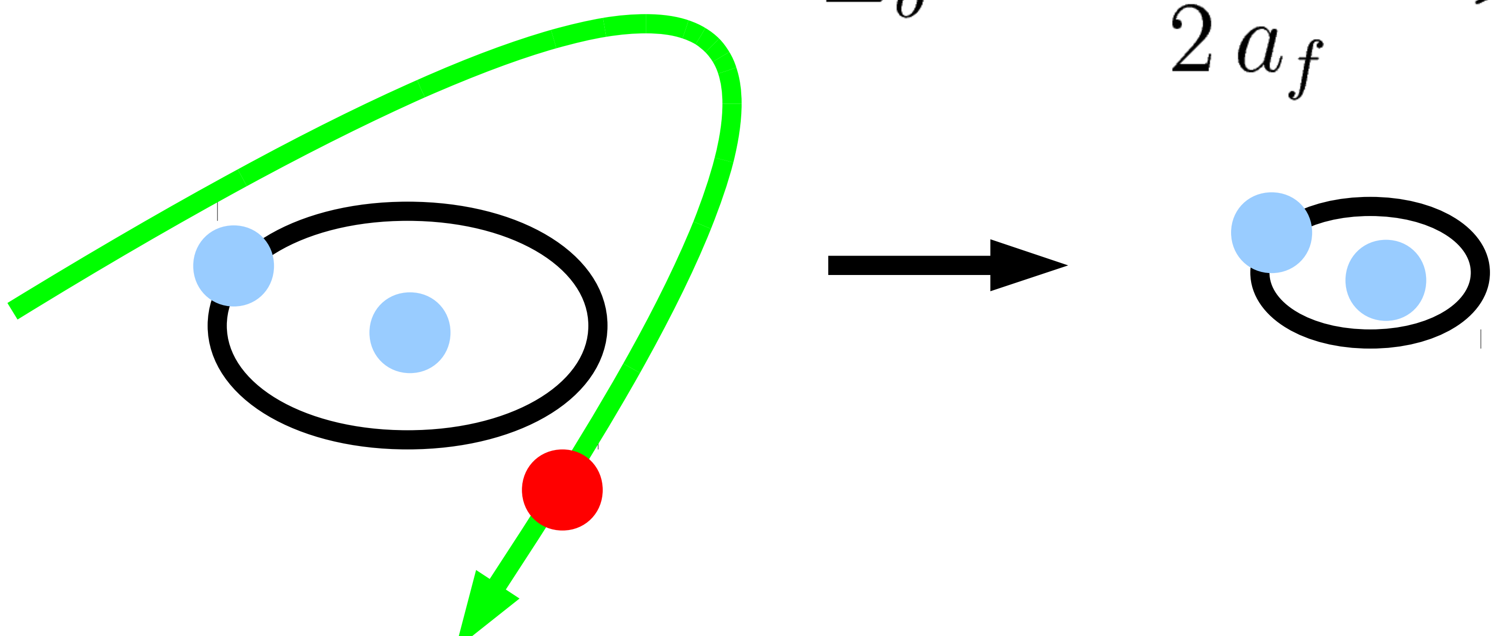
BINARIES as ENERGY RESERVOIR

If the star extracts E_{int} from the binary, its final kinetic energy (K_f) is higher than the initial kinetic energy (K_i). To better say: K_f of the centres-of-mass of the single star and of the binary is higher than their K_i.
We say that the STAR and the BINARY acquire RECOIL VELOCITY.

E_{int} becomes more negative, i.e. E_b higher: the binary becomes more bound (e.g. *a* decreases or m_1 and m_2 change).

$a_f < a_i$

$$E_b = \frac{G\,m_1\,m_2}{2\,a_f} > \frac{G\,m_1\,m_2}{2\,a_i}$$



CARTOON of a FLYBY ENCOUNTER where $a_f < a_i$ → E_b increases

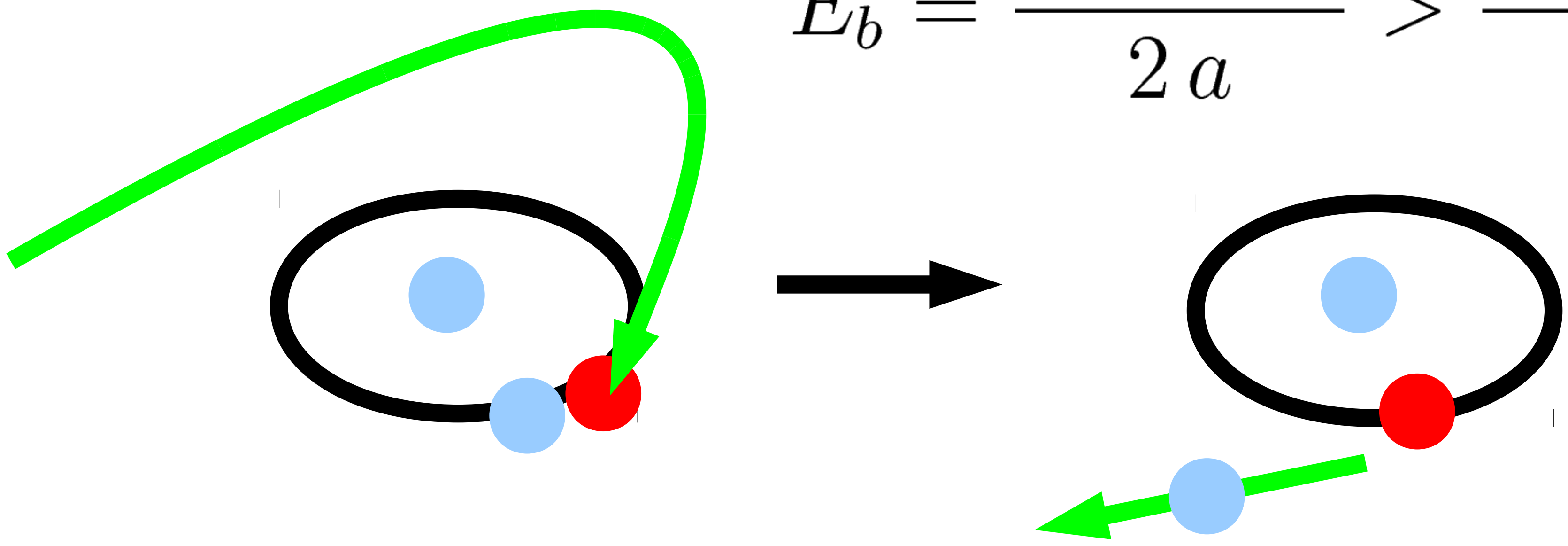
BINARIES as ENERGY RESERVOIR

An alternative way for a binary to transfer internal energy to field stars and increase its binding energy E_b is an **EXCHANGE**:
the single star replaces one of the former members of the binary.

An exchange interaction is favoured when the mass of the single star m_3 is HIGHER than the mass of one of the members of the binary so that the new E_b of the binary is higher than the former:

$m_3 > m_2$

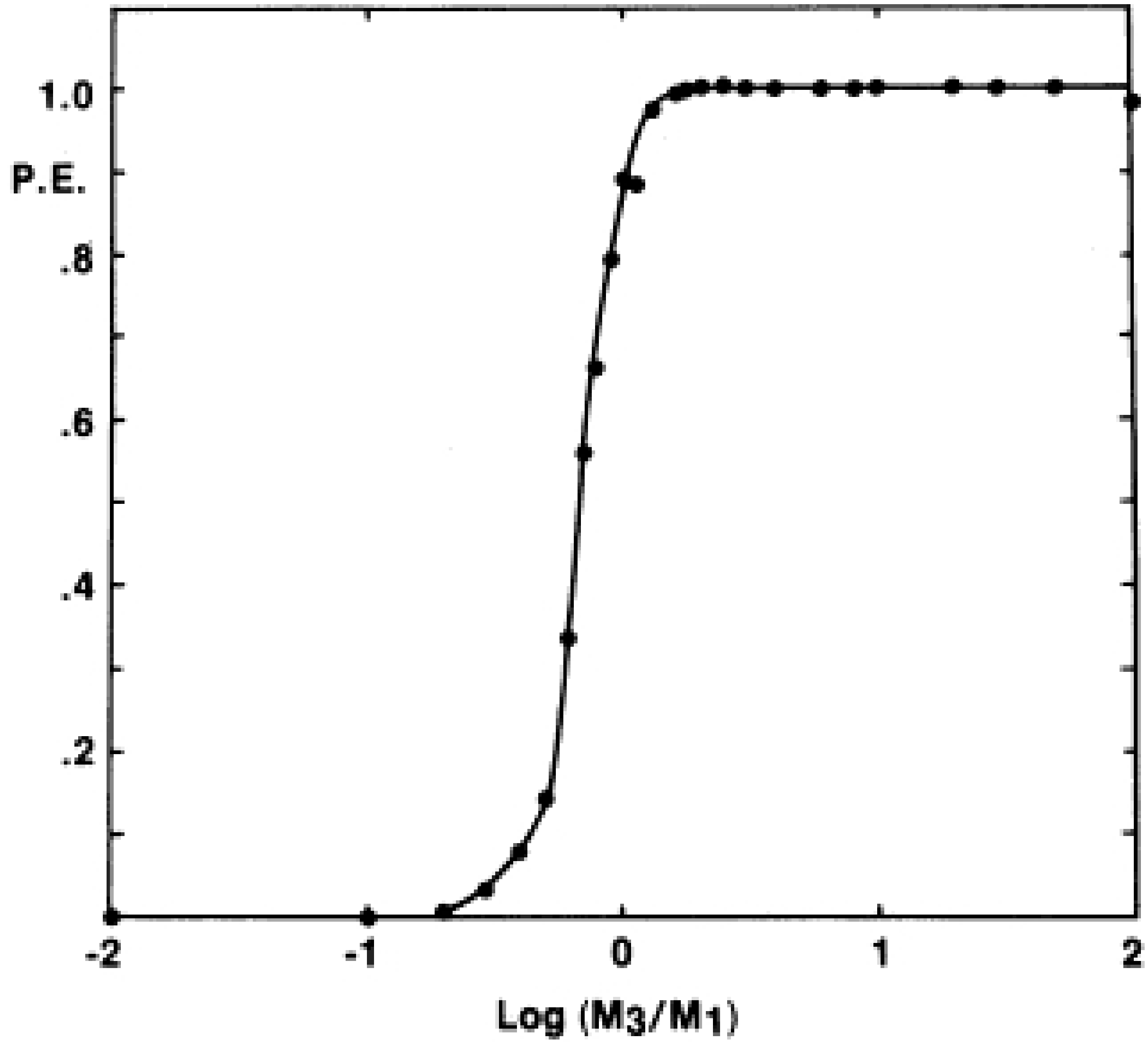
$$E_b = \frac{G\, m_1\, m_3}{2\, a} > \frac{G\, m_1\, m_2}{2\, a}$$



CARTOON of a EXCHANGE ENCOUNTER where $m_3 > m_2 \rightarrow$ E_b increases

EXCHANGE PROBABILITY

Probability increases dramatically if
$m_3 \geq m_1$



Hills & Fullerton 1980, AJ, 85, 1281

BINARIES as ENERGY RESERVOIR

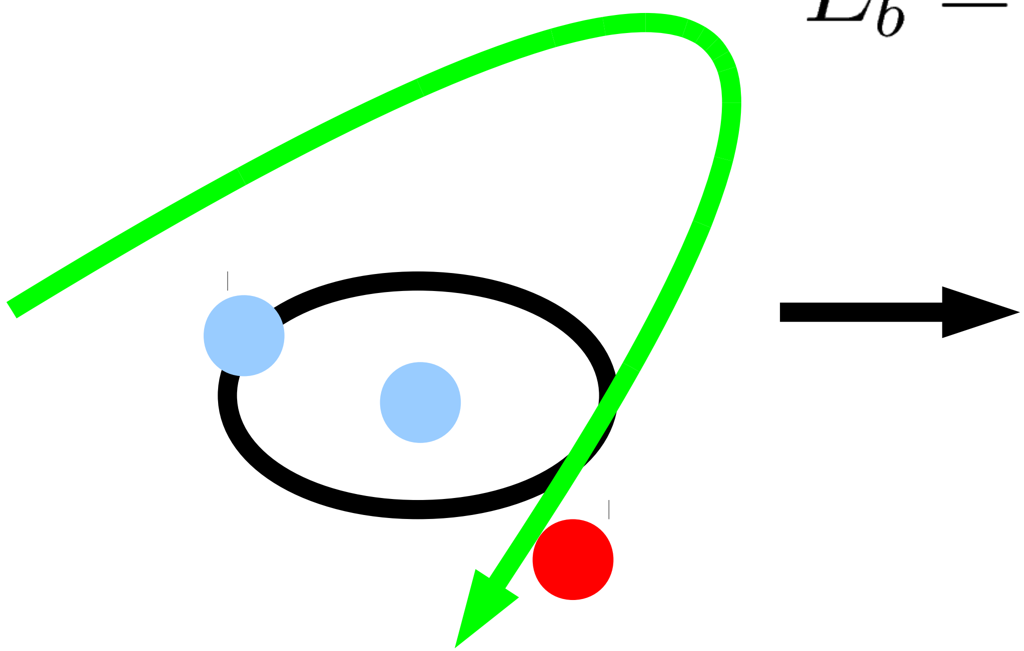
If the star transfers kinetic energy to the binary, its final kinetic energy (K_f) is obviously lower than the initial kinetic energy (K_i). To better say:

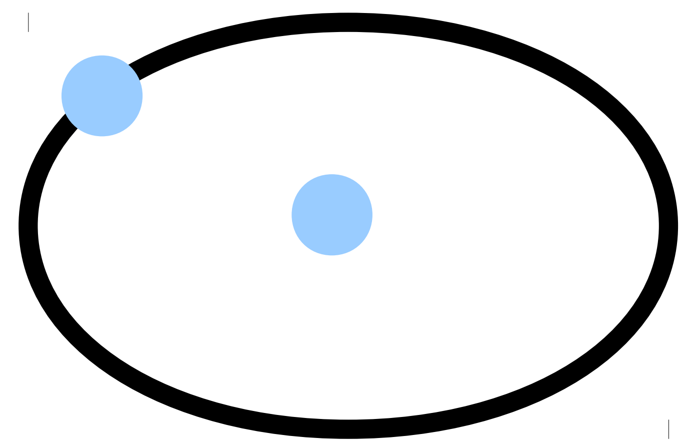
K_f of the centres-of-mass of the single star and of the binary is lower than their K_i.

E_{int} becomes less negative, i.e. E_b smaller: the binary becomes less bound (e.g. a increases) or is even IONIZED (:= becomes UNBOUND).

$a_f > a_i$

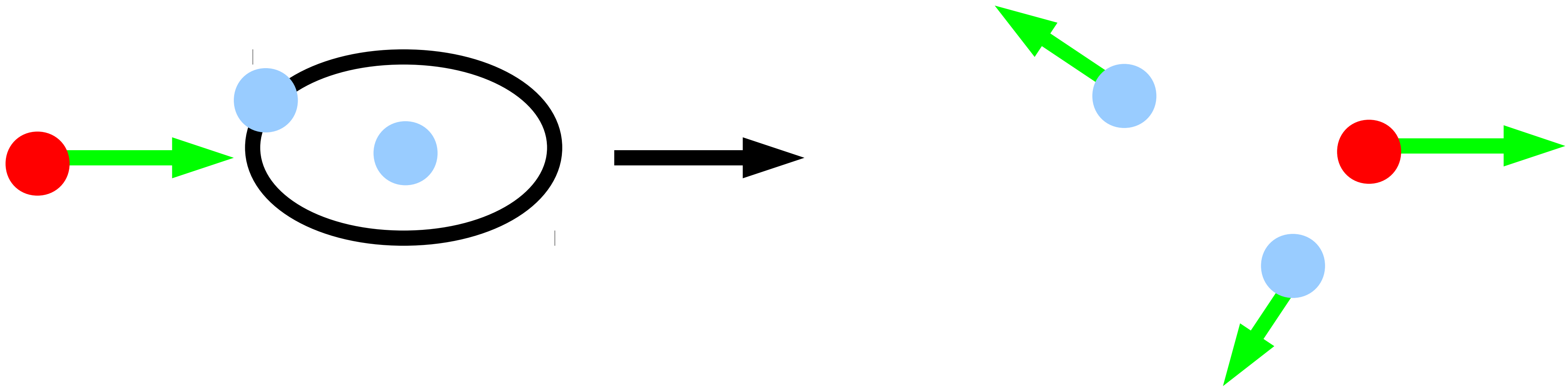
$$E_b = \frac{G\, m_1\, m_2}{2\, a_f} < \frac{G\, m_1\, m_2}{2\, a_i}$$





CARTOON of a FLYBY ENCOUNTER where $a_f > a_i \rightarrow E_b$ decreases

BINARIES as ENERGY RESERVOIR



A single star can IONIZE the binary only if its velocity at infinity (=when it is far from the binary, thus unperturbed by the binary) exceeds the CRITICAL VELOCITY (Hut & Bahcall 1983, ApJ, 268, 319)

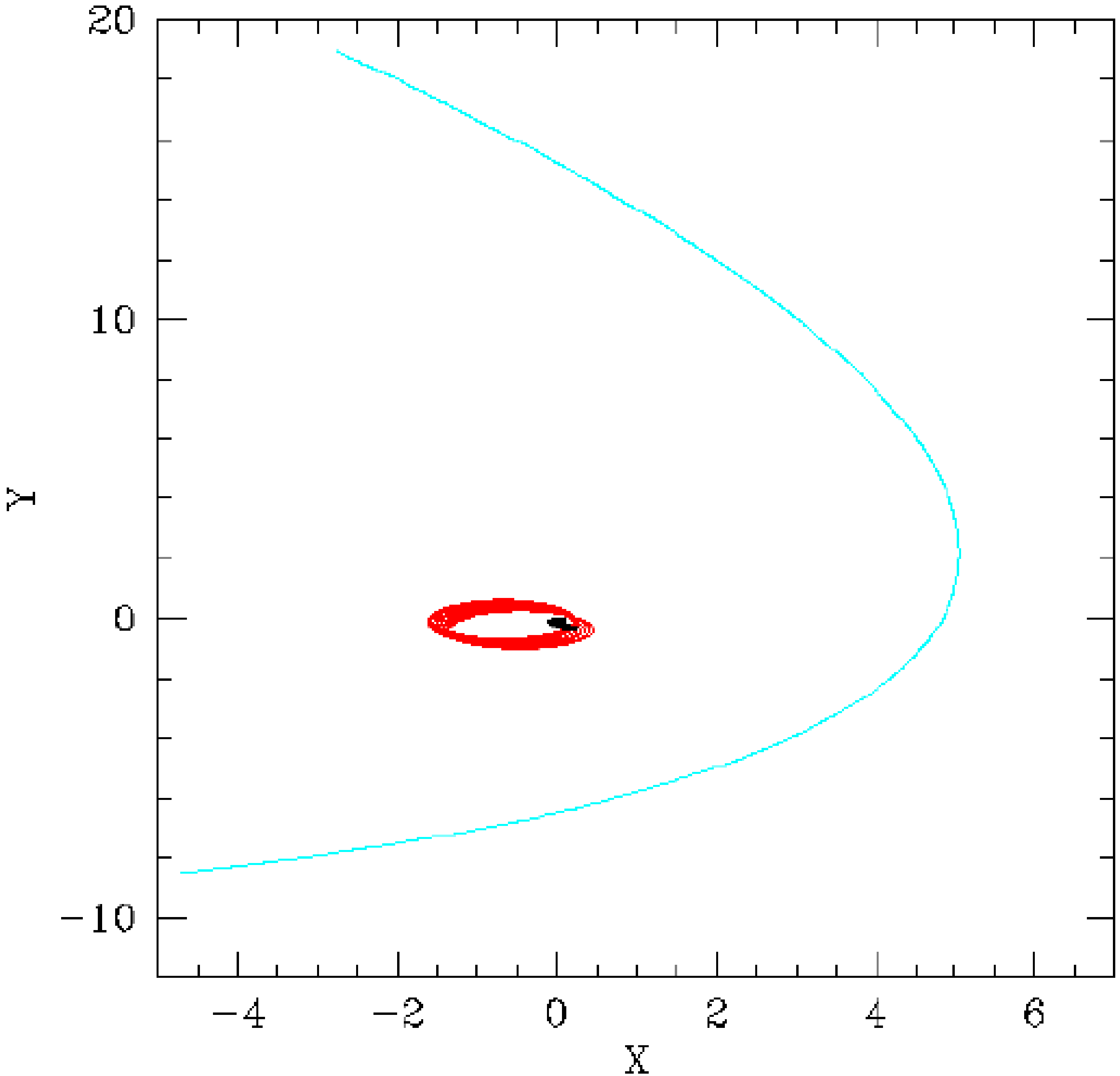
$$v_c = \sqrt{\frac{G\, m_1\, m_2\, (m_1 + m_2 + m_3)}{m_3\, (m_1 + m_2)\, a}}$$

This critical velocity was derived by imposing that the K of the reduced particle of the 3-body system is equal to E_b:

$$\frac{1}{2}\, \frac{m_3\,(m_1 + m_2)}{(m_1 + m_2 + m_3)}\, v_c^2 = \frac{G\, m_1\, m_2}{2\, a}$$

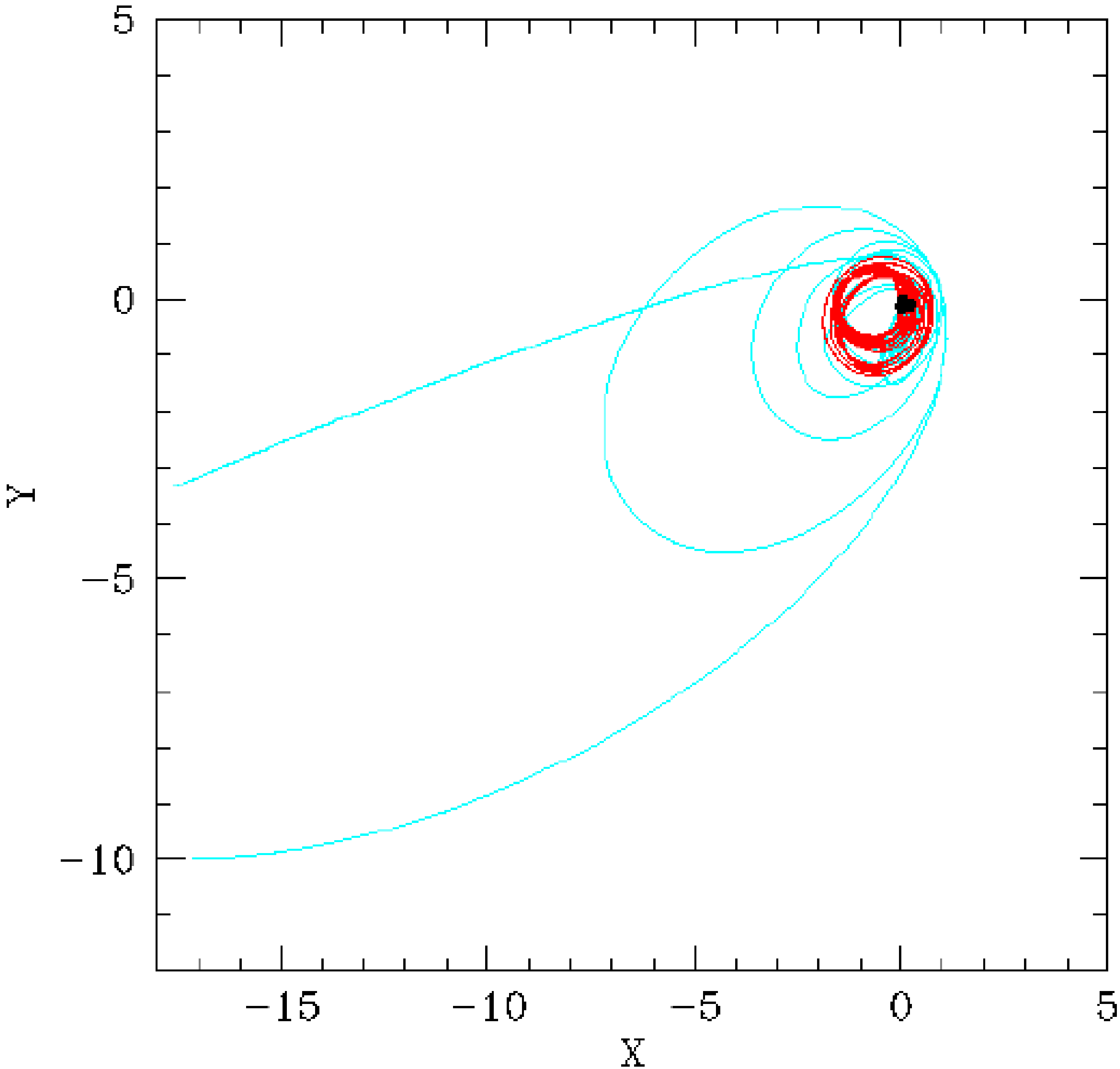
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

PROMPT
FLYBY:



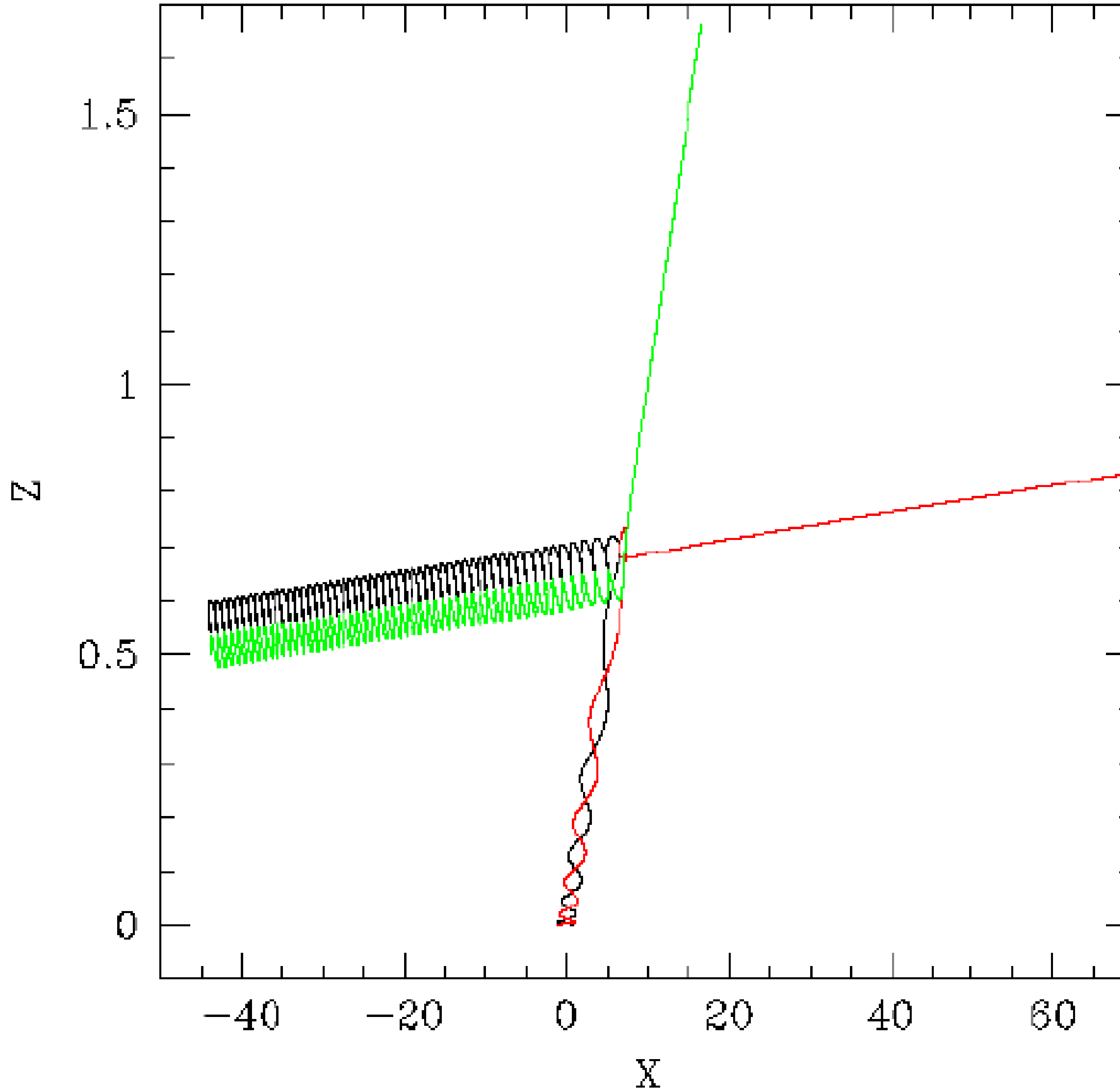
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

RESONANT
FLYBY:



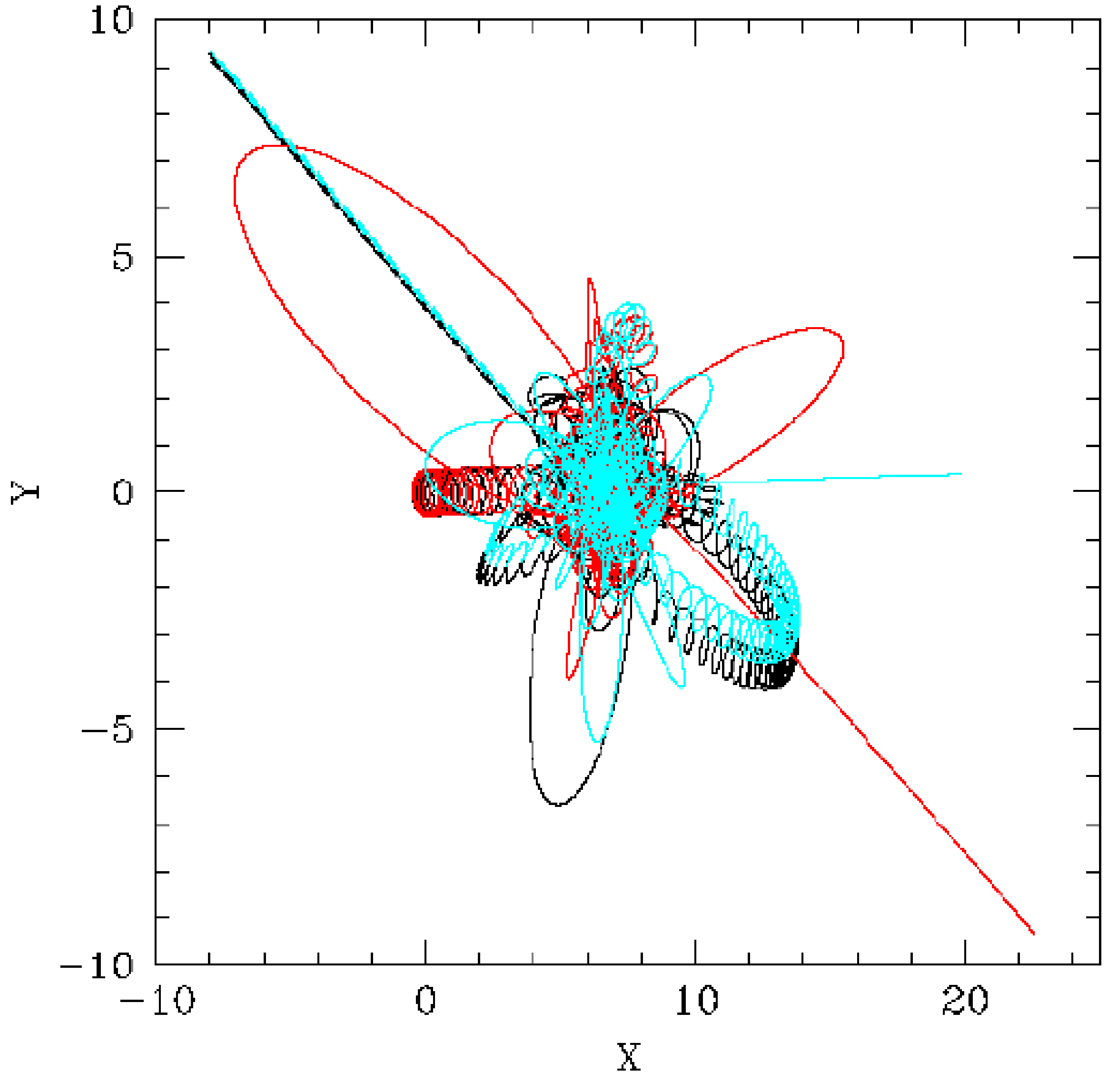
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

PROMPT
EXCHANGE:



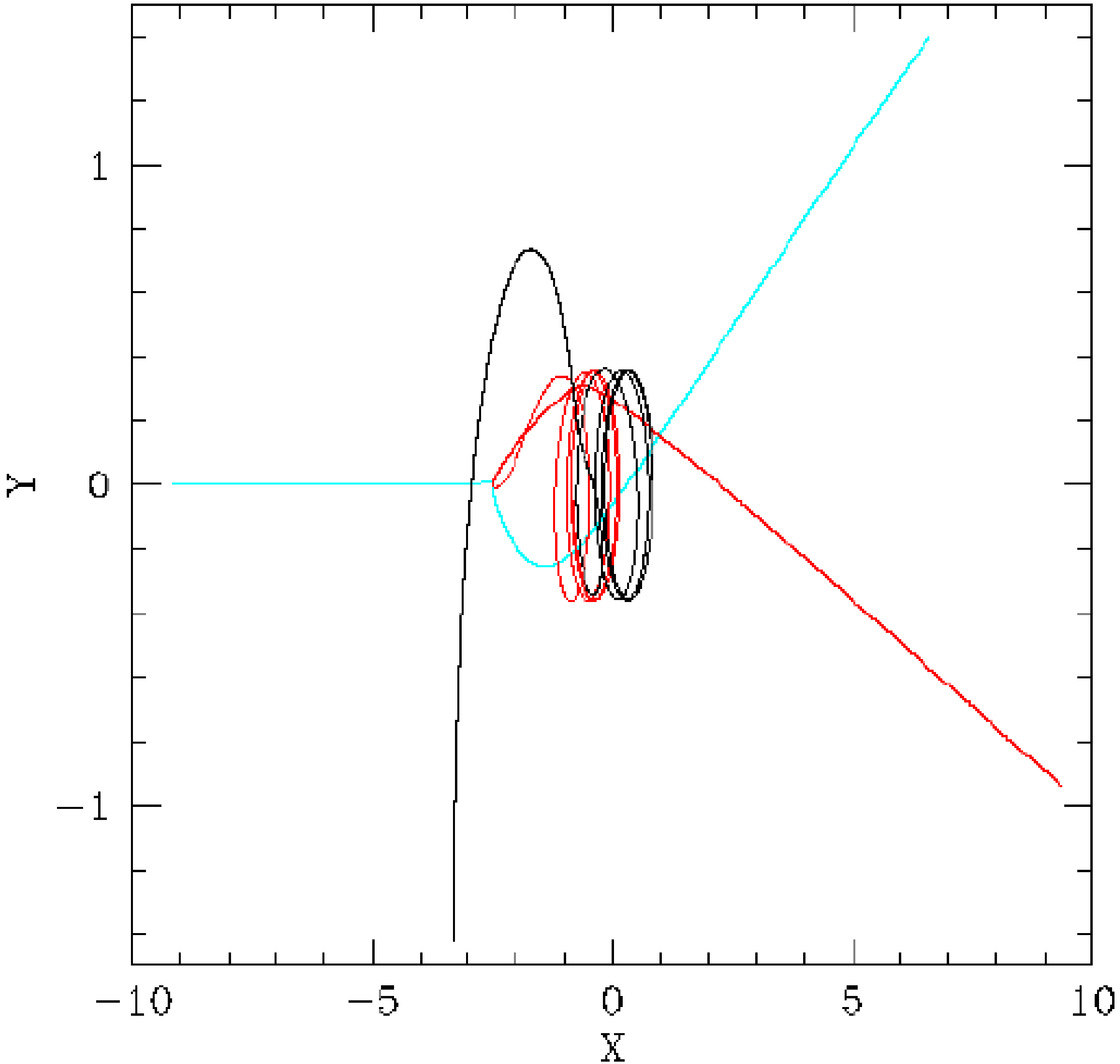
EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

RESONANT
EXCHANGE:



EXAMPLES of SIMULATED 3-BODY ENCOUNTERS

IONIZATION:



Can we understand whether a binary will lose or acquire E_b?

YES, but ONLY in a STATISTICAL SENSE

We define **HARD BINARIES:** binaries with binding energy higher than the average kinetic energy of a star in the cluster

$$\frac{G\, m_1\, m_2}{2\, a} > \frac{1}{2}\, \langle m \rangle\, \sigma^2$$

$$\frac{G\, m_1\, m_2}{2\, a} < \frac{1}{2}\, \langle m \rangle\, \sigma^2$$

SOFT BINARIES: binaries with binding energy lower than the average kinetic energy of a star in the cluster

HEGGIE'S LAW (1975):

Hard binaries tend to become harder (i.e. increase E_b)

Soft binaries tend to become softer (i.e. decrease E_b)

as effect of three-body encounters

Cross Section for 3-body encounters

Importance of binaries for dynamical encounters also due to the LARGER CROSS SECTION with respect to single stars

Simplest formalism for the cross section of stars and binaries:
GEOMETRICAL CROSS SECTION

$$\Sigma_{GE} = \pi\, a^2 \qquad\qquad \Sigma_{GE} = \pi\, R_*^{\;2}$$

For binaries (scales with a^2) For stars (scales with star radius R_*)

$a > 10^{13}$ cm, $R_* \sim 10^{10\text{-}13}$ cm $\rightarrow$ $a \gg R_*$

It is sufficient that a fraction <~ 0.1 of stars in a cluster are binaries for 3-body encounters to be more important than two-body encounters.

The difference is even larger if we take a more realistic definition of **CROSS SECTION for 3-BODY ENCOUNTERS:**

$$\Sigma = \pi\, b_{max}^2$$

b_{max} is the maximum impact parameter for a non-zero energy exchange between star and binary

Cross Section for 3-body encounters

How do we estimate b_{max}? It depends on which energy exchange we are interested in.
If we are interested only in particularly energetic exchanges, we can derive b_{max} from GRAVITATIONAL FOCUSING.

GRAVITATIONAL FOCUSING:
If the binary is significantly more massive than the single star, the TRAJECTORY of the single star is deflected by the binary, when approaching the pericentre.
The link between the impact parameter b and the effective pericentre p can be derived by the conservation of energy and angular momentum in the system of the reduced particle (see demonstration in the next slide):

$$p = \frac{G\, m_T}{{v_\infty}^2}\left[\sqrt{1+\left(\frac{{v_\infty}^2}{G\, m_T}\right)^2 b^2} - 1\right]$$

Note: in most calculations v_∞ and σ (velocity dispersion) will be used as synonymous

where $m_T \equiv m_1 + m_2 + m_3$

$$\text{if } \frac{G\, m_T}{{v_\infty}^2\, b} >> 1 \Rightarrow p \sim b^2\, \frac{v_\infty^2}{2\, G\, m_T}$$

DEMONSTRATION of GRAVITATIONAL FOCUSING FORMULA:

1) Energy conservation

$$\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_\infty^2 - \frac{G\,m_1\,m_2}{a} - \frac{G\,m_3\,(m_1+m_2)}{D} = \frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_p^2 - \frac{G\,m_1\,m_2}{a} - \frac{G\,m_3\,(m_1+m_2)}{p}$$

$D \to \infty$, $\frac{G\,m_3\,(m_1+m_2)}{D} \to 0$

$$\frac{1}{2}\,\frac{v_\infty^2}{m_T} = \frac{1}{2}\,\frac{v_p^2}{m_T} - \frac{G}{p}$$

2) Angular momentum conservation

$$\vec{J}_\infty = (m_1\,m_2)\sqrt{\frac{a_{in}\,G}{m_1+m_2}}\,\hat{z} + b\,v_\infty\left[\frac{m_3\,(m_1+m_2)}{m_T}\right]\hat{z}'$$

$\hat{z}$: parallel to *b*; $\hat{z}'$: perpendicular to *b*

$$\vec{J}_p = (m_1\,m_2)\sqrt{\frac{a_{in}\,G}{m_1+m_2}}\,\hat{z} + p\,v_p\left[\frac{m_3\,(m_1+m_2)}{m_T}\right]\hat{z}'$$

$$\Rightarrow b\,v_\infty \sim p\,v_p$$

DEMONSTRATION of GRAVITATIONAL FOCUSING FORMULA:

1) and 2) together: $$\frac{1}{2}\frac{v_\infty^2}{m_T} - \frac{1}{2}\left(\frac{b\,v_\infty}{p}\right)^2\frac{1}{m_T} + \frac{G}{p} = 0$$

Multiplying and dividing by p^2

$$\frac{1}{2}\frac{v_\infty^2}{m_T}\,p^2 + G\,p - \frac{1}{2}\frac{(b\,v_\infty)^2}{m_T} = 0$$

Finally:

$$p = \frac{-G \pm \sqrt{G^2 + \frac{b^2 {v_\infty}^4}{m_T^2}}}{{v_\infty}^2/m_T} = \frac{G\,m_T}{{v_\infty}^2}\left[\sqrt{1 + \left(\frac{{v_\infty}^2}{G\,m_T}\right)^2 b^2} - 1\right]$$

$$\text{if } \frac{G\,m_T}{{v_\infty}^2\,b} >> 1 \Rightarrow p \sim b^2\,\frac{v_\infty^2}{2\,G\,m_T}$$

Cross Section for 3-body encounters

We now express b_{max} in terms of the pericentre distance p to obtain a useful formalism for the 3-body cross section (especially for massive binaries).

$$\Sigma = \pi \left(\frac{2\, G\, m_T}{{v_\infty}^2} \right) p_{max}$$

A good choice for p_{max} (if we are interested only in the most energetic encounters) is $p_{max}=a$

$$\boxed{\Sigma = 2\, \pi\, G\, \frac{m_T\, a}{{v_\infty}^2}} \qquad (*)$$

There are many other formalisms for the cross section. We remind the one in Davies (2002):

$$\Sigma = \kappa(q_1, q_2)\, \pi a^2 \left[1 + \left(\frac{v_c}{v_\infty} \right)^2 \right]$$

3-body interaction rate

As usual, an interaction rate has the form

$$R = \frac{dN}{dt} = n\,\Sigma\,v_\infty$$

where *n* is the local density of stars.
For the cross section in (*), the rate becomes

$$\boxed{R = 2\,\pi\,G\,\frac{m_T\,n\,a}{v_\infty}}$$

Note: in most calculations v_∞ and σ (velocity dispersion) will be used as synonymous

Note: the rate depends

(1) on the total mass of the interacting objects (more massive objects interact more),
(2) on the semi-major axis of the binary (wider binaries have a larger cross section),
(3) on the local density (denser environments have higher interaction rate),
(4) on the local velocity field (systems with smaller velocity dispersion have higher interaction rate).

Energy exchanges

Most general formalism (from Energy conservation):

$$-\frac{G\,m_1\,m_2}{a_i}+\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_\infty^2=-\frac{G\,m_a\,m_b}{a_f}+\frac{1}{2}\,\frac{m_e\,(m_a+m_b)}{m_T}\,v_{fin}^2 \qquad (+)$$

m_a, m_b and m_e are the final mass of the primary binary member, the final mass of the secondary binary member and the final mass of the single star, respectively (these may be different from the initial ones in the case of an exchange).

Change in binding energy:

$$\Delta E_b=\frac{1}{2}\,G\left(\frac{m_a\,m_b}{a_f}-\frac{m_1\,m_2}{a_i}\right)$$

$$\frac{\Delta E_b}{E_{b,\,in}}=\left(\frac{m_a\,m_b}{m_1\,m_2}\,\frac{a_i}{a_f}-1\right)$$

If NO EXCHANGE (m_a=m_1, m_b=m_2)

$$\frac{\Delta E_b}{E_{b,\,in}}=\left(\frac{a_i}{a_f}-1\right)$$

If $a_i / a_f > 1$ the quantity ΔE_b is transferred to the kinetic energy of the involved centres of mass

SUPERELASTIC ENCOUNTERS: kinetic energy increases after interaction, because the binary is source of additional energy

Energy exchanges

Energy exchanges can be approximately QUANTIFIED if

1) binary is hard
2) p is <~ 2a
3) mass of the single star is small respect to binary mass
(exchanges are unlikely)

If 1), 2) and 3) hold, simulations show that $$\frac{\Delta E_b}{E_b} \propto \frac{m_3}{m_1 + m_2}$$

Hills (1983, AJ, 88, 1269) defines the post-encounter energy parameter

$$\xi \equiv \frac{m_1 + m_2}{m_3} \frac{\langle \Delta E_b \rangle}{E_b}$$

Where $<\Delta E_b>$ is the average binding energy variation per encounter.

ξ can be extracted from N-body simulations:

*depends slightly on binary eccentricity (ξ~2 if e=0, ξ~6 if e=0.99)

*depends slightly on binary mass ratio (ξ~2 if $m_1=m_2$, ξ~4 if m_1/m_2=10-30)

* depends strongly on impact parameter (a factor of >200 between b=0 and b=20a)

by averaging over relevant impact parameters **ξ=0.2 – 1**

From E_b definition

$$\langle \Delta E_b \rangle = \xi \frac{m_3}{m_1 + m_2} \frac{G\, m_1\, m_2}{2\, a}$$

Hardening rate

Rate of binding energy exchange for a hard binary

$$\frac{dE_b}{dt} = \langle \Delta E_b \rangle \frac{dN}{dt} = \xi \frac{m_3}{m_1 + m_2} E_b \frac{dN}{dt}$$

Where *dN/dt* is the collision rate. Using formalism in slide 19

$$\frac{dE_b}{dt} = \xi \frac{\langle m \rangle}{m_1 + m_2} E_b \frac{2\,\pi\, G\,(m_1 + m_2)\, n\, a}{\sigma}$$

Average star mass (because average energy exchange)

Note: **$\langle m \rangle\, n = \rho$** (local mass density of stars)

$$\frac{dE_b}{dt} = \pi\, \xi\, G^2 \frac{\rho}{\sigma} m_1\, m_2$$

Note: in most calculations v_∞ and σ (velocity dispersion) will be used as synonymous

**Depends only on cluster environment and binary mass!*

**Constant in time, if the cluster properties do not change and if the binary members do not exchange:*

–> *'A hard binary hardens at a constant rate'* (Heggie 1975, 3-body Bible)

Hardening rate

Expressing *a* in terms of E_b (assuming m_1 and m_2 constant, i.e. no exchange)

$$\frac{d}{dt}\left(\frac{1}{a}\right) = \frac{2}{G\,m_1\,m_2}\,\frac{dE_b}{dt} = 2\,\pi\,G\,\xi\,\frac{\rho}{\sigma}$$

Also called **HARDENING RATE**

From it we can derive the average time evolution of the semi-major axis of a hard binary:

$$\frac{da}{dt} = -2\,\pi\,G\,\xi\,\frac{\rho}{\sigma}\,a^2$$

It means that the smaller *a*, the more difficult is for the binary to shrink further (because cross section becomes smaller).

When binary is very hard, three body encounters are no longer efficient: further evolution of the binary is affected by tidal forces and merger (if it is composed of soft bodies) or by gravitational wave emission (if the two binary members are compact objects). When? See further discussion on timescales.

Relevant timescales

1) HARDENING TIMESCALE

$$t_h = \left| \frac{a}{\dot{a}} \right| = \frac{\sigma}{2\,\pi\,G\,\xi}\,\frac{1}{a}$$

2) GRAVITATIONAL WAVE (GW) TIMESCALE

For a binary of compact objects it is important to know whether the main driver of orbital evolution is hardening or GW decay.
From Peters (1964, Gravitational radiation and the motion of two point masses, Phys. Rev. B136, 1224) the timescale of orbital decay by GWs is

$$t_{GW} = \frac{5}{256}\,\frac{c^5\,a^4\,\left(1-e^2\right)^{7/2}}{G^3\,m_1\,m_2\,(m_1+m_2)}$$

Combining 1) and 2) we can find the maximum semi-major axis for GWs to dominate evolution

$$a_{GW} = \left[\frac{256}{5}\,\frac{G^2\,m_1\,m_2\,(m_1+m_2)\,\sigma}{2\,\pi\xi\,\left(1-e^2\right)^{7/2}\,c^5\,\rho}\right]^{1/5}$$

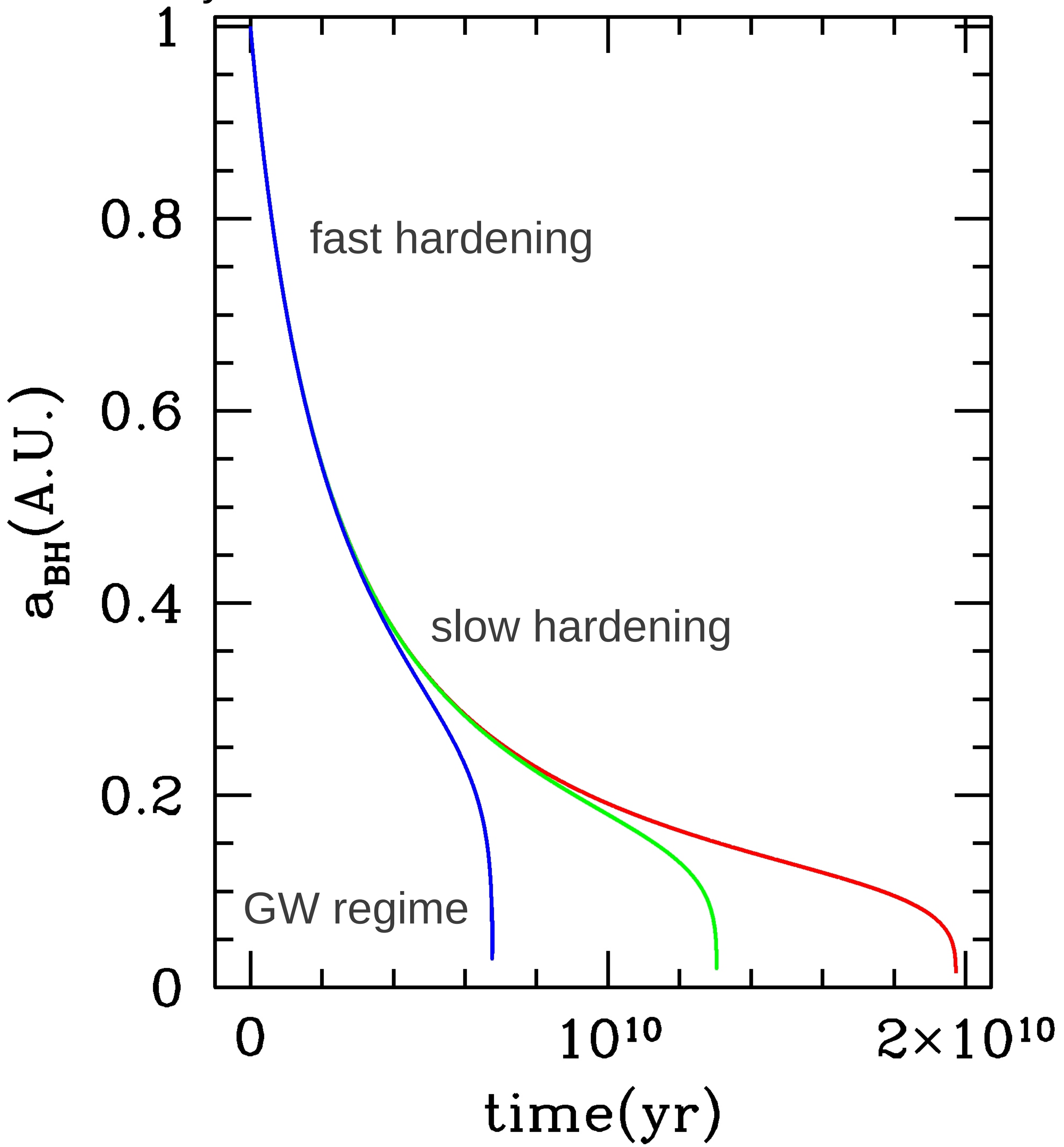
Relevant timescales

Example: black hole – black hole binary

* blue
  m_1=200 $M_\odot$
  m_2=10 $M_\odot$

* green
  m_1=50 $M_\odot$
  m_2=10 $M_\odot$

* red
  m_1=30 $M_\odot$
  m_2=3 $M_\odot$



Number of encounters before GW regime

$$N_{int} = \int_0^t R\,dt = \int_0^t \frac{2\,\pi\,G\,m_T\,n\,a}{\sigma}\,dt$$

$$= \int_{a_0}^{a(t)} \frac{2\,\pi\,G\,m_T\,n\,a}{\sigma}\,\frac{\sigma\,da}{-2\,\pi G\,\xi\rho\,a^2} = \int_{a(t)}^{a_0} \frac{1}{\xi}\,\frac{m_T}{\langle m\rangle}\,\frac{da}{a}$$

↑

$$\frac{da}{dt} = -2\,\pi\,G\,\xi\,\frac{\rho}{\sigma}\,a^2$$

$$= \frac{1}{\xi}\,\frac{m_T}{\langle m\rangle}\,\ln\left(\frac{a_0}{a(t)}\right)$$

Relevant timescales

1) INTERACTION TIMESCALE (from interaction rate)

$$t_{3b} = \frac{\sigma}{2\,\pi\,G\,m_T\,n\,a}$$

2) DYNAMICAL FRICTION TIMESCALE

$$t_{\rm df} = \frac{3}{4\,(2\,\pi)^{1/2}\,G^2\,\ln\Lambda}\,\frac{\sigma^3}{m_T\,\rho(r)}$$

If $t_{df} \ll t_{3b}$, dynamical friction washes velocity changes induced by 3-body encounters (especially for very small *a*) and especially out of core (where σ drops)

If $t_{df} \gg t_{3b}$, 3-body encounters dominate the binary velocity (especially for large *a*)

Recoil velocities

Most general expression of recoil velocity for the reduced particle (Sigurdsson & Phinney 1993)

$$v_{fin} = \sqrt{\frac{m_3\,(m_1+m_2)}{m_e\,(m_a+m_b)}\,v_\infty^2 + \frac{2\,m_T}{m_e\,(m_a+m_b)}\,\Delta E_b}$$

m_a, m_b and m_e are the final mass of the primary binary member, the final mass of the secondary binary member and the final mass of the single star, respectively (these may be different from the initial ones in the case of an exchange).

This equation comes from (+) at slide 20:

$$\frac{1}{2}\,\frac{m_3\,(m_1+m_2)}{m_T}\,v_\infty^2 + \Delta E_b = \frac{1}{2}\,\frac{m_e\,(m_a+m_b)}{m_T}\,v_{fin}^2$$

What happens to the binary, then?

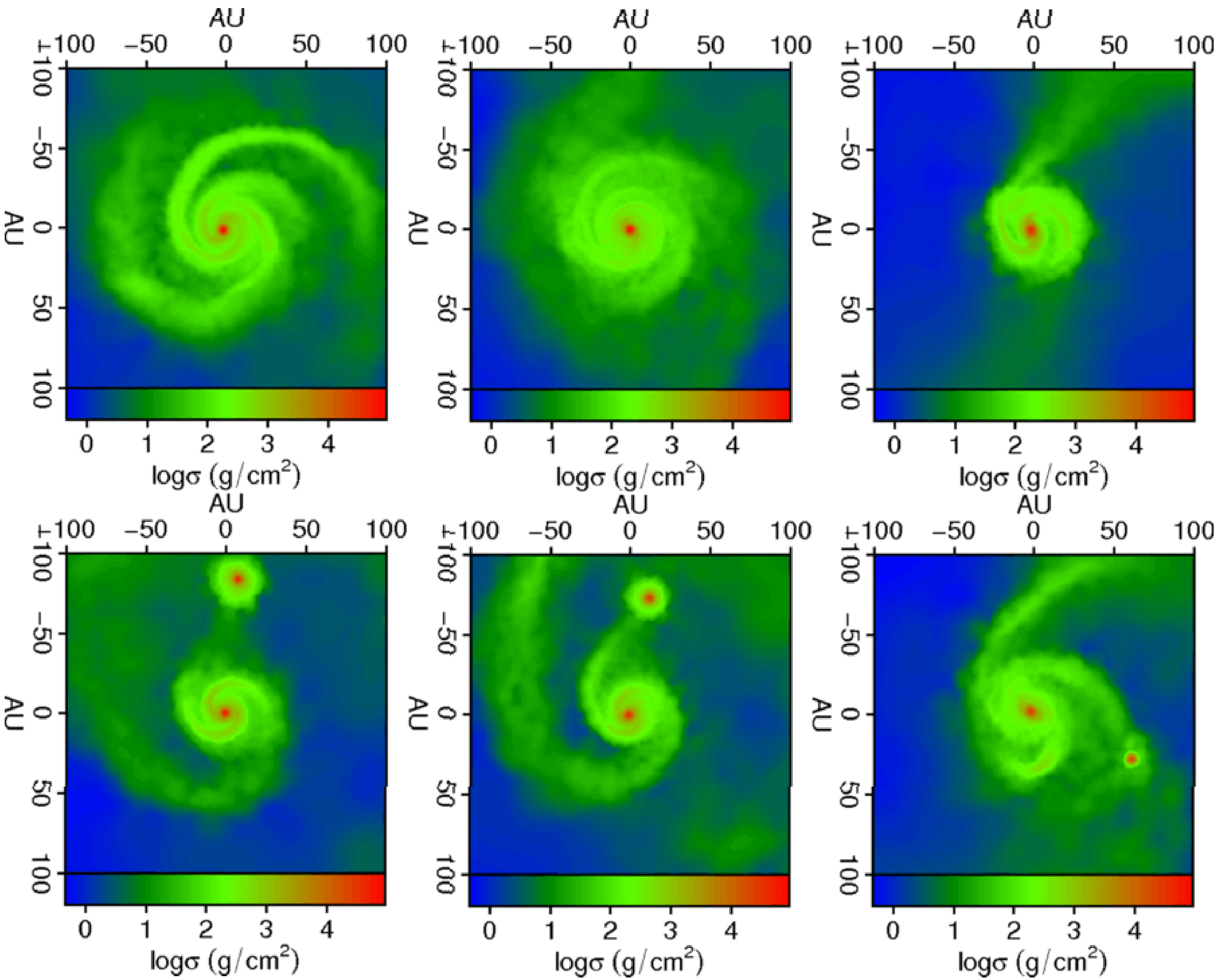
The recoil of the binary (if the binary is more massive than the single star -i.e. the motion of the single star coincides almost with that of the reduced particles) follows from conservation of linear momentum

$$v_{rec} = \frac{m_e}{m_T}\,v_{fin}$$

Origin of binaries

Which are the formation pathways of binaries?

1) primordial binaries: binaries form from the same accreting clump in the parent molecular cloud (very difficult to understand with simulations)



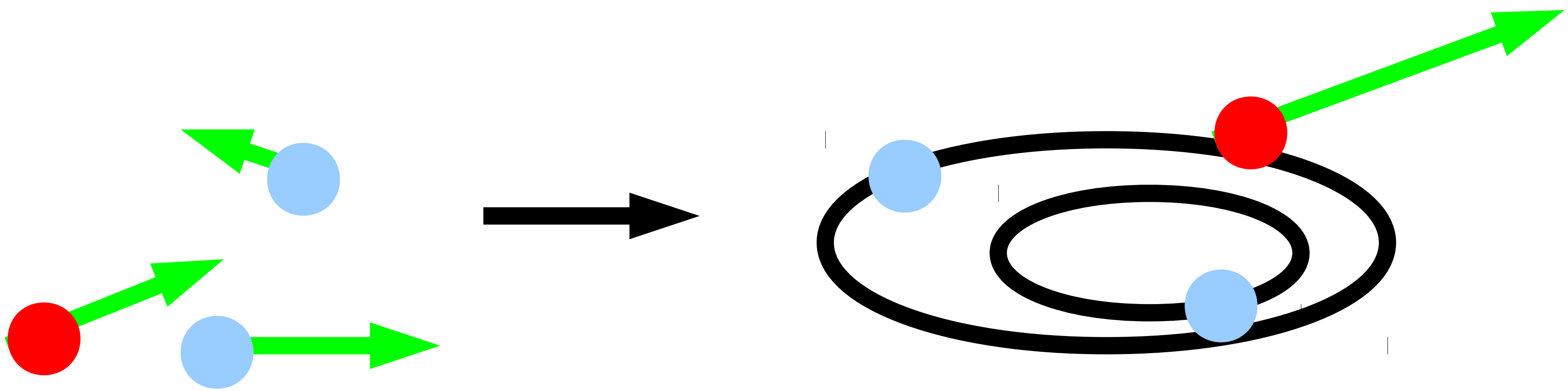
SPH simulation of a single star and a binary formed from a molecular cloud

Hayfield et al. 2011, MNRAS, 2011, 417, 1839

Origin of binaries

Which are the formation pathways of binaries?

2) three-body induced binaries: 3 single stars pass close to each other
And one of the three brings away sufficient energy to leave the others bound (only soft-ish binaries with high eccentricity). Unlikely unless high density (core collapse)



$$\frac{dn_b}{dt} = 1.91 \times 10^{-13} \left(\frac{n}{10^4\,\mathrm{pc}^3}\right)^3 \left(\frac{m}{m_\odot}\right)^5 \left(\frac{10\,\mathrm{km\ s}^1}{\langle v \rangle}\right)^9 \mathrm{pc}^{-3}\,\mathrm{yr}^{-1}$$

Origin of binaries

Which are the formation pathways of binaries?

3) tidally induced binaries: two stars pass very close to each other so that they feel each other tidal field → energy dissipation produces a bound couple (very hard binaries with ~0 eccentricity for dissipation – merge in most cases). Unlikely unless high density (core collapse)

Tidal radius (from Roche limit):

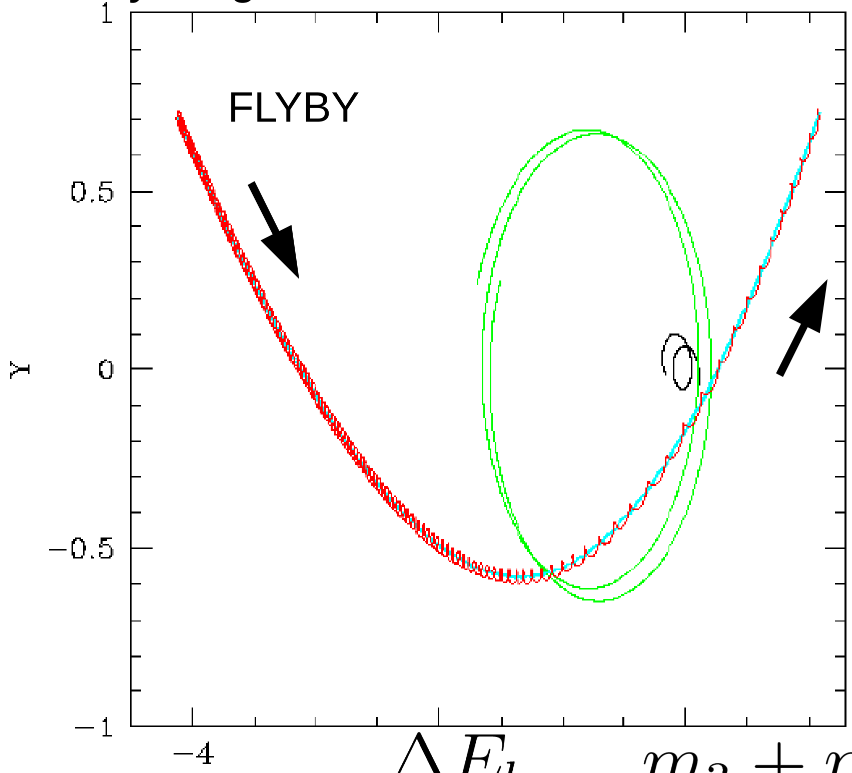
$$r_t = r_2 \left(\frac{2\, m_1}{m_2} \right)^{1/3}$$

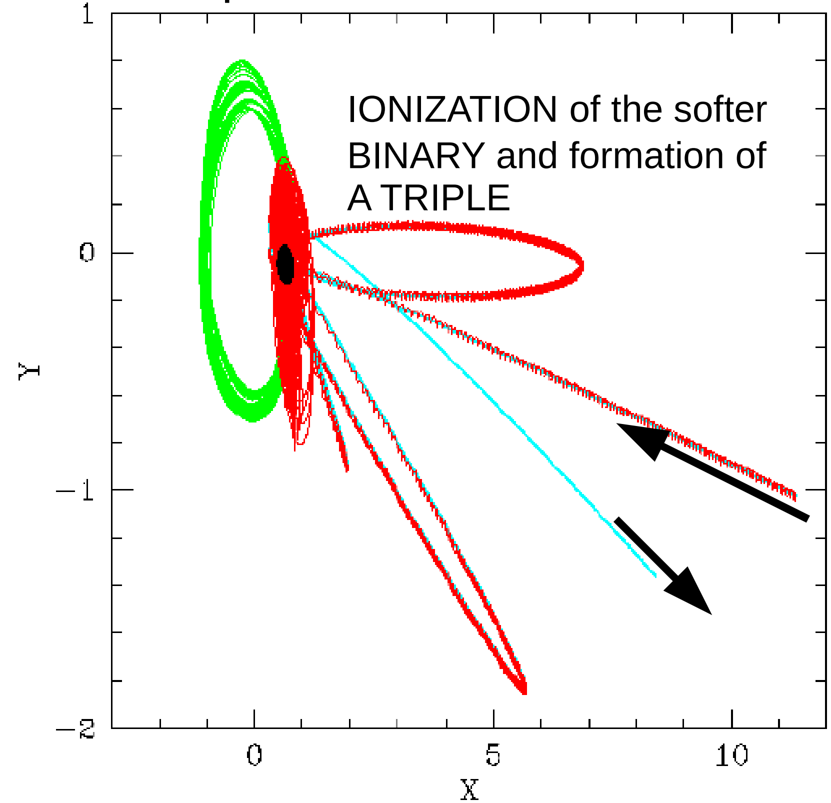
valid if $m_1 > m_2$

4) binaries formed by exchange: in general, they are not considered new binaries as they come from a pre-existing binary after exchange of members

4- (5-, 6-, …) body encounters

Everything we said still holds, but with higher level of complications





Energy exchanges: $$\frac{\Delta E_b}{E_b} \propto \frac{m_3 + m_4}{m_1 + m_2}$$

Critical velocity for ioniz. $$v_c = \sqrt{\frac{G\,m_T}{(m_1+m_2)\,(m_3+m_4)}\left(\frac{m_1\,m_2}{a_1} + \frac{m_3\,m_4}{a_2}\right)}$$

Even RARE STABLE TRIPLEs can form

(Mardling & Aarseth 1999 stability criterion)

r_p, e_{ou}= pericentre and eccentricity of the outer binary

a_{in}= semi-major axis of the inner binary

$q=m_{ou}/m_{in}$ mass ratio (outer to inner binary)

$$\frac{r_p}{a_{in}} \geq 2.8\left[(1+q)\,\frac{1+e_{ou}}{\sqrt{1-e_{ou}^2}}\right]^{2/5}$$

Role of binaries in cluster evolution

1*Density increase during core collapse HARDENS primordial binaries and ENHANCES FORMATION of binaries by encounters of three single stars and by tidal captures

2*Tidally formed binaries are too small: only a few interactions or merger → negligible for energy transfer

3*HARD primordial binaries and new 3-body formed binaries REVERSE core collapse by transferring their internal energy to *K* of stars in core VIA 3-BODY ENCOUNTERS

4*stars that undergo 3-body encounters with hard binaries are ejected ($|W|$ decreases) or remain in the core and transfer *K* to other stars

5**Note: HARD binaries transfer K to the core but SOFT binaries extract K from the core. Why are hard binaries predominant?*
TOTAL $|E_i|$ *of HARD BINARIES >> TOTAL* $|E_i|$ *of SOFT BINARIES!!!!*

6*Mass losses by stellar winds and supernovae can help, but only if the timescale for massive star evolution is ~ core collapse time!!! (see last lecture)

7*If there are >1 very hard binaries might eject each other by 4-body encounters

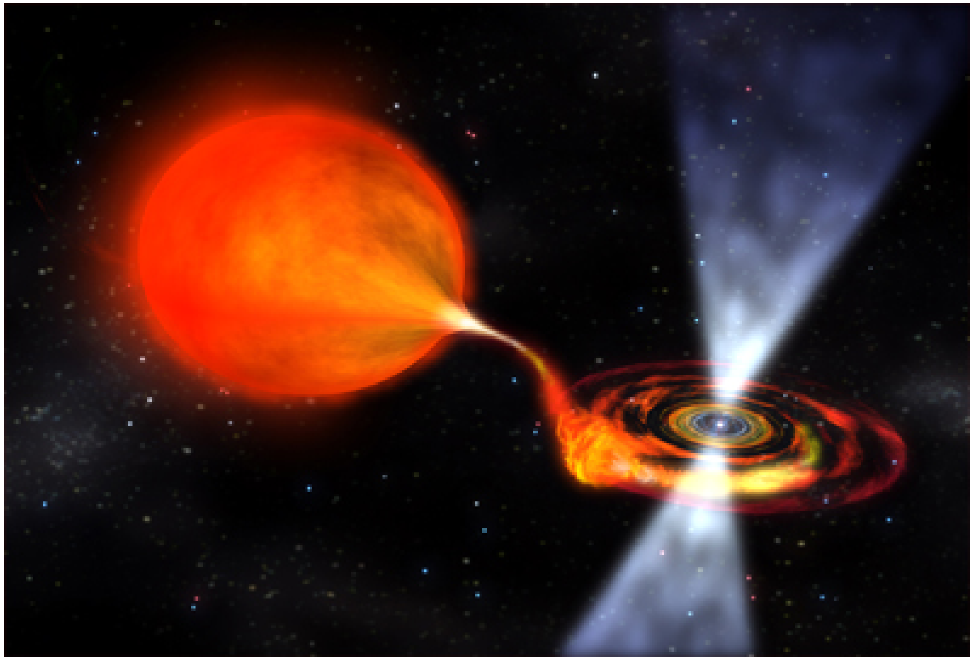
Role of binaries in formation of exotica

Binaries and three-body encounters are the main suspects for the formation of STELLAR EXOTICA, such as

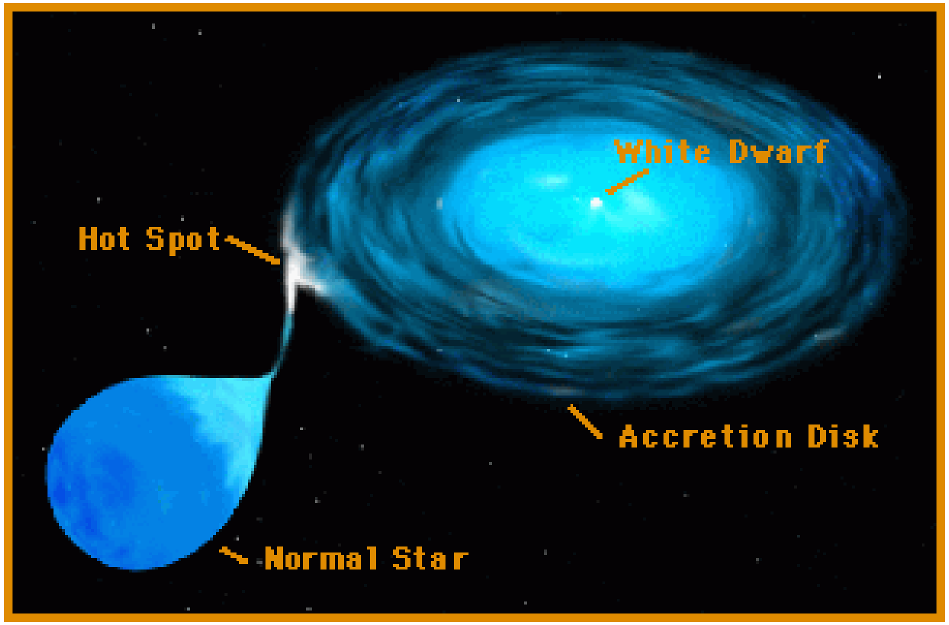
* blue straggler stars

* massive BHs and intermediate-mass BHs

* millisecond pulsars



* cataclismic variables



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