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LECTURES on COLLISIONAL DYNAMICS:

2. STAR CLUSTER EVOLUTION

GRANULARITY of the GRAVITATIONAL FIELD

Interactions between 2 stars (**two-body encounters**)
PRODUCE LOCAL FLUCTUATIONS of ENERGY BALANCE i.e.
CHANGE LOCALLY THE MAGNITUDE of STELLAR VELOCITIES

On the RELAXATION TIMESCALE, this induces
GLOBAL CHANGES in the CLUSTER EQUILIBRIUM

HOW?

Very simple approach based on probability distribution:

Relaxation leads to a Maxwellian (thermal) velocity distribution

$\wp$:= probability of finding a star within an energy interval ΔE at a given energy E (for Maxwellian velocity distribution)

$$\wp \propto \mathrm{EXP}(-\mathrm{B}E)\,\Delta E$$

$\Rightarrow$ stars tend to have $E \ll 0$ (because $\wp$ is higher)

⇒ most bound stars (central core) become more bound

⇒ central CORE CONTRACTS

BUT the TOTAL ENERGY of cluster must remain ~ CONSTANT (cluster self-bound and isolated)

⇒ less bound stars (halo stars) absorb kinetic energy released by most bound stars (core stars)

⇒ less bound stars become less bound

⇒ HALO EXPANDS (stars move to higher E or become unbound) to keep E_{TOT}=constant

With virial theorem:

$$\mathbf{0 = W + 2\,K}$$

$$\text{if } W \downarrow \Rightarrow K \uparrow$$

IN DETAIL: there are at least 3 physical processes that determine cluster evolution

(1) EVAPORATION:

escape of stars in the high velocity tail of the Maxwellian

$$f\,(v \rightarrow \infty) > 0$$

(2) GRAVOTHERMAL INSTABILITY:

Instability which occurs in a small core confined in outer ISOTHERMAL halo

(3) EQUIPARTITION:

stars in a cluster tend to have the same average kinetic energy

(1) if equal mass stars → the same average velocity

(2) if bodies in the system have different mass

$$m_i \langle v_i^2 \rangle = m_j \langle v_j^2 \rangle$$

$$\text{if} \quad m_i > m_j \Rightarrow \langle v_i^2 \rangle < \langle v_j^2 \rangle$$

(1) EVAPORATION:

Escape velocity of a star from a cluster: $$\frac{1}{2}\,v_e^2 = |\phi|$$

where ϕ= potential
as the kinetic energy of
the star must overcome its potential energy

MEAN SQUARE escape velocity of a star from a cluster:

$$\langle v_e^2 \rangle = \frac{\int \rho(r)\, v_e^2(r)\, dV}{\int \rho(r)\, dV} = \frac{\int \rho(r)\, 2\, |\phi(r)|\, dV}{M} = -4\frac{W}{M}$$

$$W = \frac{1}{2}\int \rho(r)\, \phi(r) dV$$

from virial theorem:

$$\langle v_e^2 \rangle = 4\,\frac{2\,K}{M} = 4\,\langle v^2 \rangle$$

a star can escape if its velocity is higher than 2 times the root mean square velocity

(1) EVAPORATION:

The concept of evaporation is simple: if $v>v_e \Rightarrow$ the star escapes

We add a mathematical model to understand the evolution of the system induced by evaporation in the case of

CONSTANT RATE OF MASS LOSS PER UNIT MASS PER TIME INTERVAL dt / t_{rlx}

This assumption implies self-similarity, as the radial variation of density, potential and other quantities are time-invariant except for
TIME DEPENDENT SCALE FACTORS
Example: a contracting uniform sphere which remains uniform (:= density independent of radius) during contraction

⇒ **MASS LOSS RATE:**

$$\frac{dM}{dt} = \frac{-\xi_e M(t)}{t_{rlx}(t)} = -\frac{\xi_e M(0)}{t_{rlx}(0)} \left[\frac{R(t)}{R(0)}\right]^{-3/2} \left[\frac{M(t)}{M(0)}\right]^{1/2}$$

CONSTANT RATE OF MASS LOSS

where we used the fact that

$$M(t) = \frac{M(0)}{M(0)} M(t)$$

$$t_{rlx}(t) = t_{rlx}(0) \left(\frac{R(t)}{R(0)}\right)^{3/2} \left(\frac{M(t)}{M(0)}\right)^{1/2}$$

(1) EVAPORATION:

Previous equation has two unknowns (*M(t), R(t)*) → we need another equation:
Change of total cluster energy, as each escaping star carries away a certain **kinetic energy per unit mass (=ζ E_m, where E_m is the mean energy per unit mass of the cluster)**

$$\frac{dE_{TOT}}{dt} = \zeta\, E_m \frac{dM}{dt} = \frac{\zeta\, E_{TOT}}{M} \frac{dM}{dt}$$

Since $E_{TOT} \propto - M^2/R$

$$\frac{dE_{TOT}}{dt} = -\frac{d}{dt}\left(\frac{M^2}{R}\right) = -\frac{2\,M}{R}\frac{dM}{dt} + \frac{M^2}{R^2}\frac{dR}{dt}$$

$$\frac{\zeta\, E_{TOT}}{M}\frac{dM}{dt} = -\zeta \frac{M}{R}\frac{dM}{dt}$$

$$(2-\zeta)\frac{dM}{M} = \frac{dR}{R} \quad \Rightarrow \quad \frac{R}{R(0)} = \left[\frac{M}{M(0)}\right]^{2-\zeta}$$

(1) EVAPORATION:

Inserting

$$\frac{R}{R(0)} = \left[\frac{M}{M(0)}\right]^{2-\zeta} \qquad \text{(@)}$$

into the equation for mass loss rate, i.e.

$$\frac{dM}{dt} = \frac{-\xi_e\, M(t)}{t_{rlx}(t)} = -\frac{\xi_e\, M(0)}{t_{rlx}(0)} \left[\frac{R(t)}{R(0)}\right]^{-3/2} \left[\frac{M(t)}{M(0)}\right]^{1/2}$$

we find:

$$\frac{dM}{dt} = -\frac{\xi_e\, M(0)}{t_{rlx}(0)} \left[\frac{M}{M(0)}\right]^{(-5+\zeta)/2}$$

Integrating the above equation:

$$\frac{M}{M(0)} = \left[1 - \frac{\xi_e\,(7-3\,\zeta)}{2}\,\frac{t}{t_{rlx}(0)}\right]^{2/(7-3\,\zeta)} \equiv \left(1 - \frac{t}{t_{coll}}\right)^{2/(7-3\,\zeta)}$$

t_{coll} := collapse time, time at which *M* and *R* vanish.

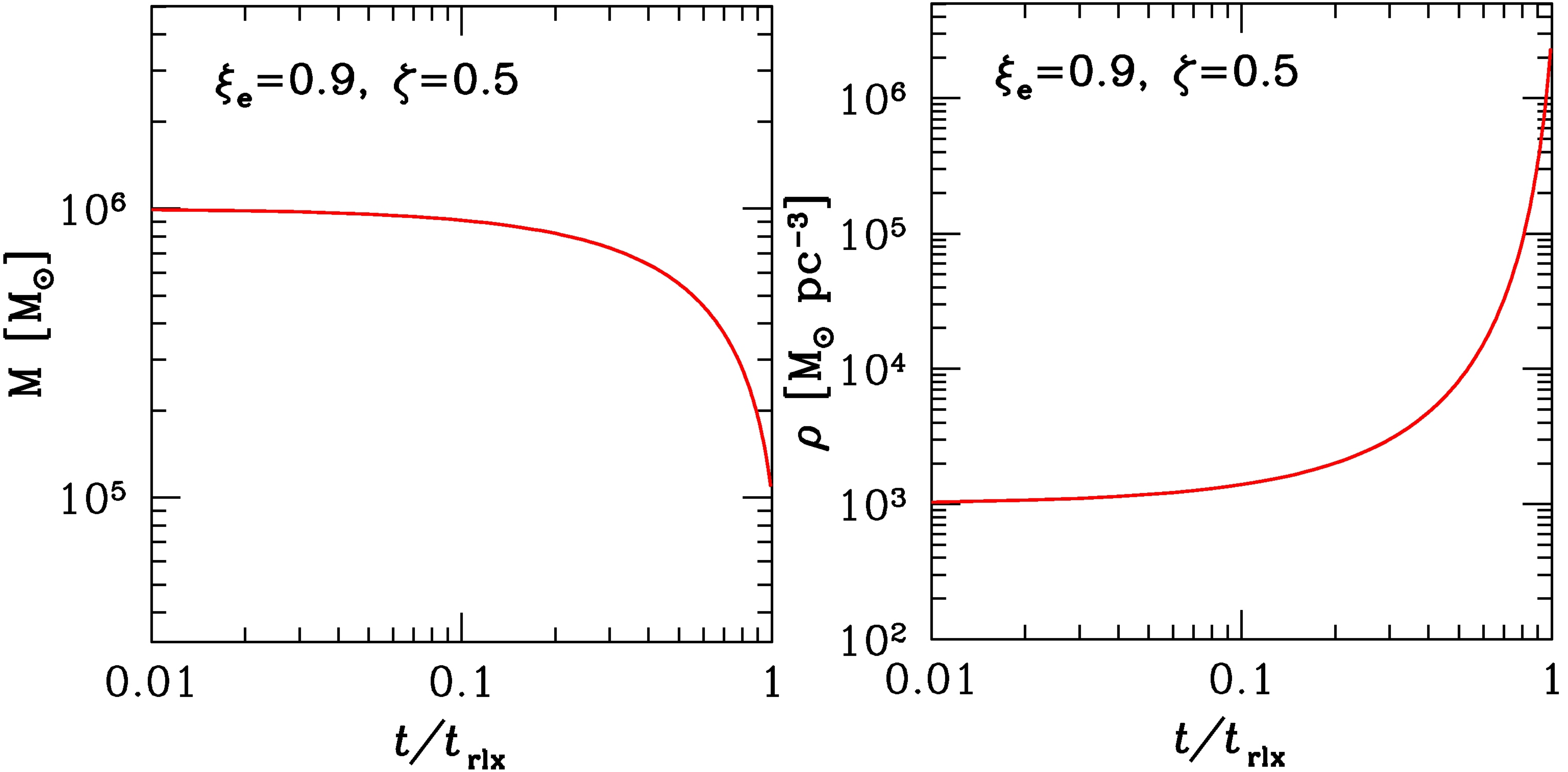
(1) EVAPORATION:

Note: from (@), using the fact that $\rho = 3\,M\,/\,(\,4\,\pi\,R^3\,)$

$$\frac{\rho}{\rho(0)} \propto \left[\frac{M}{M(0)}\right]^{-(5-3\,\zeta)}$$

Since $\zeta < 1$ (for realistic clusters), when M decreases for evaporation, ρ increases:

Collapse! Evaporation may induce collapse!!



(2) GRAVOTHERMAL INSTABILITY or CORE COLLAPSE:

Instability which occurs in a small core
confined in outer ISOTHERMAL halo
GRAV. INST. even if **STARS ARE EQUAL MASS!!!!!!**

1. MATHEMATICAL APPROACH
2. PHYSICAL APPROACH

1. MATHEMATICAL APPROACH:

analogy with IDEAL GAS

We define the temperature T of a self-gravitating system

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} \kappa_B T$$

Total kinetic energy of a system

$$K = \sum \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} \kappa_B N \frac{\int_V \rho(\mathbf{x}) T(\mathbf{x}) d\mathbf{x}}{\int_V \rho(\mathbf{x}) d\mathbf{x}}$$

(2) GRAVOTHERMAL INSTABILITY or CORE COLLAPSE:

Virial theorem: $$E_{\mathrm{TOT}} = -K = -\frac{3}{2} N \kappa_B \langle T \rangle$$

Definition of heat capacity: $$C \equiv \frac{dE}{d\langle T \rangle} = -\frac{3}{2} N \kappa_B$$

always negative

MEANS THAT by LOSING ENERGY THE SYSTEM BECOMES HOTTER

→ **system contracts more and becomes hotter in runaway sense**

How?

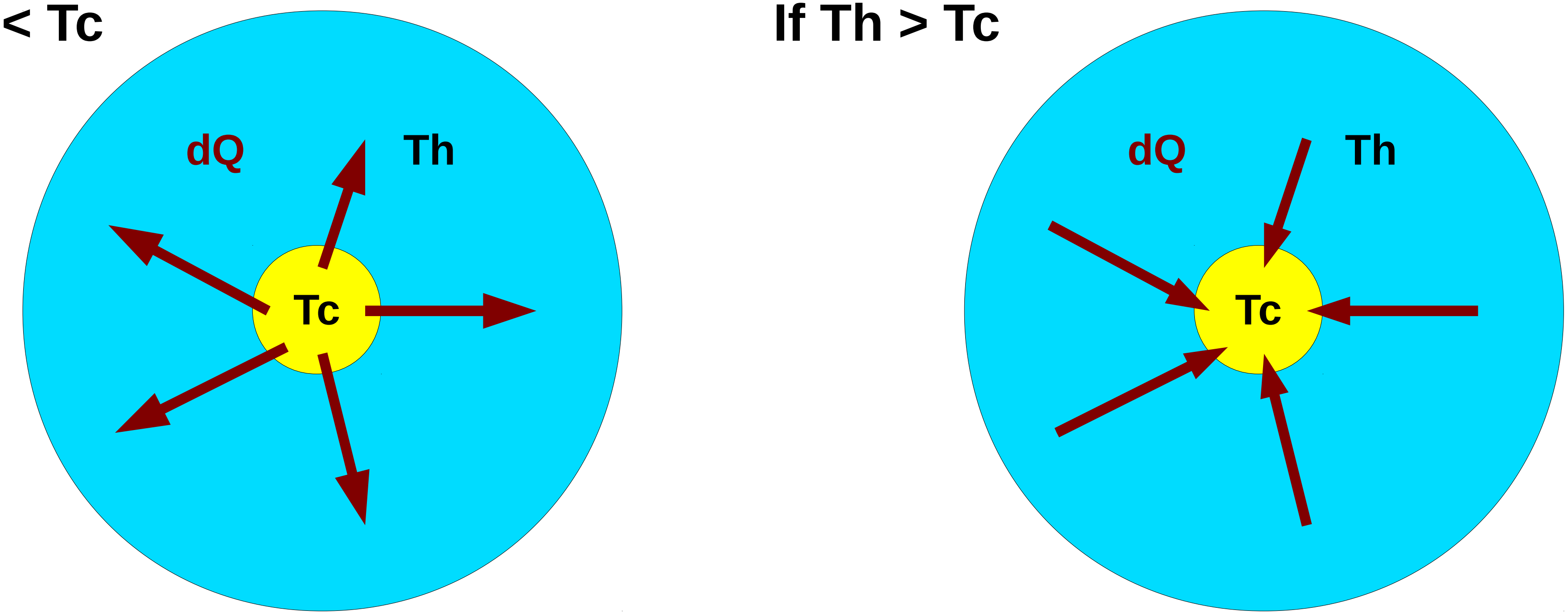
If we put a negative heat capacity system in a bath and heat is transferred to the bath (**d*Q* = *dE* > 0**)

→ temperature of the system changes by $T - \frac{dQ}{C} > T$

→ system becomes hotter and heat keeps flowing from system to bath: *T* rises without limits!!

Note: Any bound finite system in which dominant force is gravity exhibits *C*<0

If Th < Tc **If Th > Tc**



Note: CONDITION that HALO is LARGE with respect to the core is crucial!
so that K continuously injected into the halo does not imply the heating of the halo

Otherwise, if the *K* of the halo overcomes the *K* of the core,
The energy injected into the halo FLOWS BACK to the core and stops contraction!!!

(2) GRAVOTHERMAL INSTABILITY or CORE COLLAPSE:

2. PHYSICAL APPROACH:

INITIAL CONDITIONS:

* SMALL HIGH-DENSITY CORE in a very LARGE ISOTHERMAL HALO (the bath)
* MAXWELLIAN VELOCITY DISTRIBUTION (or, in general, velocity distribution where stars can evaporate)

$$f(v) \propto v^2 \, e^{-v^2} \qquad f(v) > 0 \quad \text{if } v \to \infty$$

IF Maxwellian velocity distribution

⇒ high velocity tail of stars ESCAPE from the core into halo (EVAPORATION)

⇒ $$K = \sum_i \frac{1}{2} m_i \, v_i^2$$ ↓ because high velocity stars escape

$$W = -\sum_{i,j,i\neq j} \frac{G \, m_i \, m_j}{r_{ij}}$$ ↑ because the mass of escaping stars is lost

BUT DECREASE in *K* is more important than increase in *W* since the FASTER STARS LEAVE the cluster!

$$2\,K_f + W_f < 2K_i + W_i$$

⇒ GRAVITY is NO longer supported by *K*, by random motions

⇒ SYSTEM CONTRACTS (*)

⇒ TO REACH NEW VIRIAL EQUILIBRIUM AVERAGE VELOCITY MUST INCREASE

Or, to say it in a different way (more physical?)

(*) IF the system contracts, it becomes DENSER

⇒ higher density implies MORE two-body encounters (higher two-body encounter rate)

⇒ stars exchange more energy and become dynamically hotter

⇒ faster stars tend to EVAPORATE even more than before

⇒ *K* decreases faster than *W* increases

⇒ system contracts even more

⇒ CATASTROPHE!!!

WHAT DOES REVERSE THE CORE COLLAPSE??

ONLY SWITCHING ON A NEW SOURCE OF K = K_{ext}

THIS SOURCE CAN OPERATE IN TWO WAYS

⇒ (1) $$2\,K_f + W_f = 2\,(K_{ext} + K_i) + W_i > 2\,K_i + W_i$$

Kinetic energy increases not from gravitational contraction but from an **EXTERNAL** SOURCE (breaks virial equilibrium and negative heat capacity)

→ CORE EXPANDS (lasts only till energy source is on)

⇒ (2) **THE NEW KINETIC ENERGY TRANSFERRED TO CORE STARS INDUCES THE EJECTION OF STARS THAT WERE NOT NECESSARILY THE FASTER STARS BEFORE RECEIVING THE NEW KINETIC ENERGY**:

$$K = \sum_i \frac{1}{2} m_i\, v_i^2$$ ↓ because stars which received external kinetic energy escape

$$W = -\sum_{i,j,i\neq j} \frac{G\, m_i\, m_j}{r_{ij}}$$ ↑ because the mass of escaping stars is lost

BUT INCREASE in *W* (DECREASE OF |*W*|) is more important than decrease in *K* since

(I) STARS that LEAVE the cluster were not the faster before receiving the kick and

(II) since K_f is the sum of K_i and K_{ext}!

$$2\,K_f + W_f > 2K_i + W_i$$

⇒ **POTENTIAL WELL BECOMES PERMANENTLY SHALLOWER**

⇒ **AND SYSTEM EXPANDS** (*)

⇒ TO REACH NEW VIRIAL EQUILIBRIUM AVERAGE VELOCITY MUST DECREASE

Or, to say it in a different way (more physical?)

(*) IF the system expands, it becomes LESS DENSE

⇒ lower density implies LESS two-body encounters (lower two-body encounter rate)

⇒ stars exchange less energy and become dynamically cooler

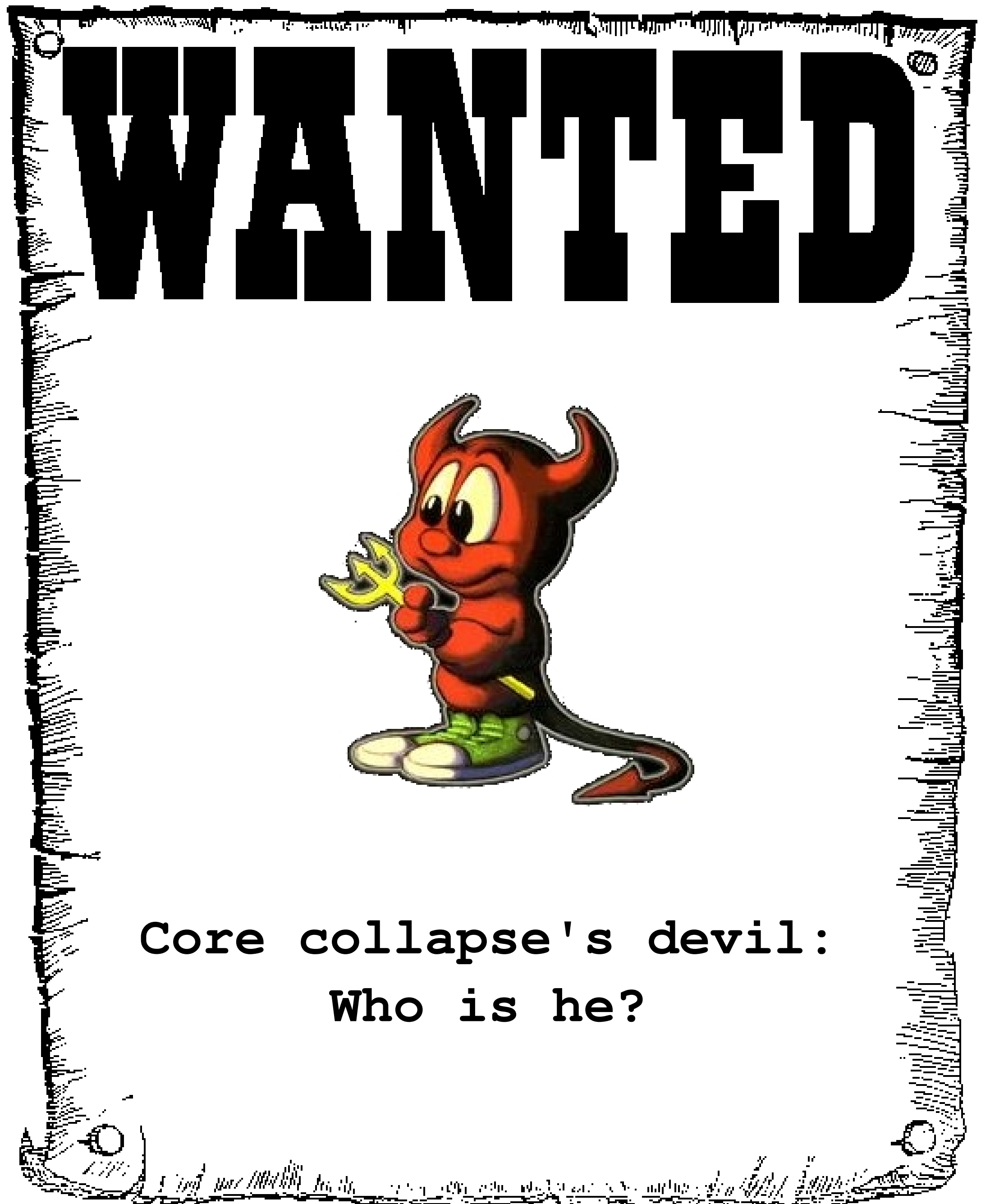
⇒ gravitational CATASTROPHE is reversed !!!

Even if sources of heating (partially) switch off, the ejection of stars and the lowering of potential well ensures reversal of catastrophe (but see gravothermal oscillations at end of lecture)

WHAT DOES REVERSE THE CATASTROPHE??

We did not make any assumption about the source of kinetic energy that reverses core collapse

BUT WHICH ARE THE MOST LIKELY SOURCES IN REAL-LIFE CLUSTERS?



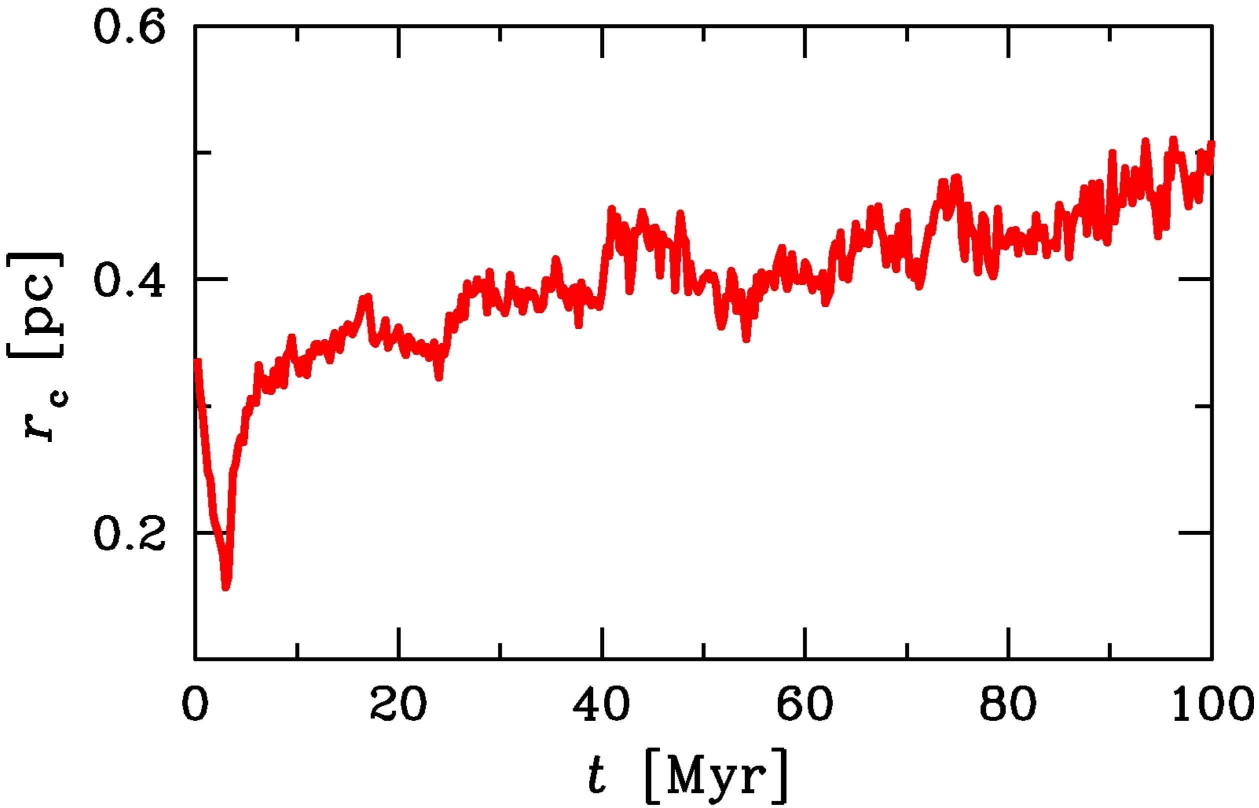
BUT WHICH IS THE NEW SOURCE OF K ENERGY WHICH SWITCHES ON?

(1) MASS LOSS by STELLAR WINDS and SUPERNOVAE which remove mass without changing *K* of other stars

$$2\,K_i + W_f > 2\,K_i + W_i$$

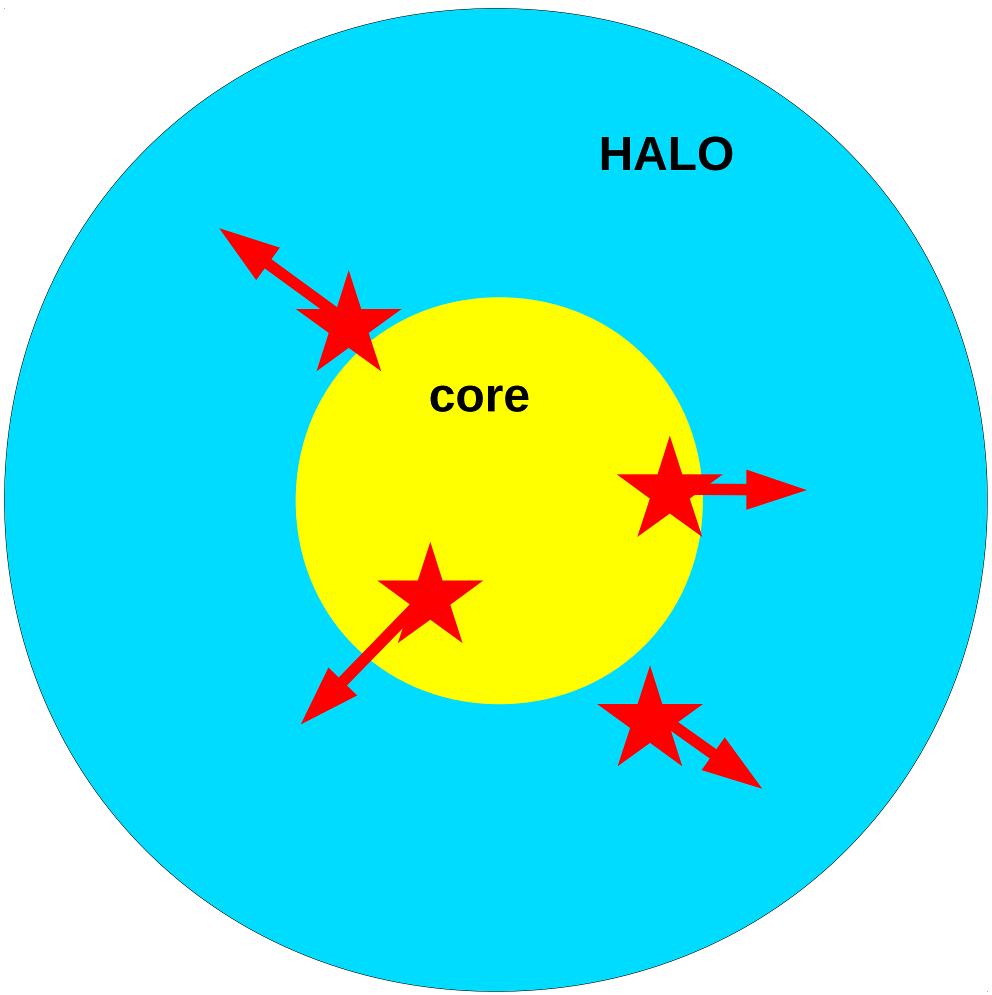
IMPORTANT only if massive star evolution lifetime is similar to core collapse timescale (see last lecture)

(2) BINARIES as ENERGY RESERVOIR (see next lecture)



CORE COLLAPSE with CARTOONS

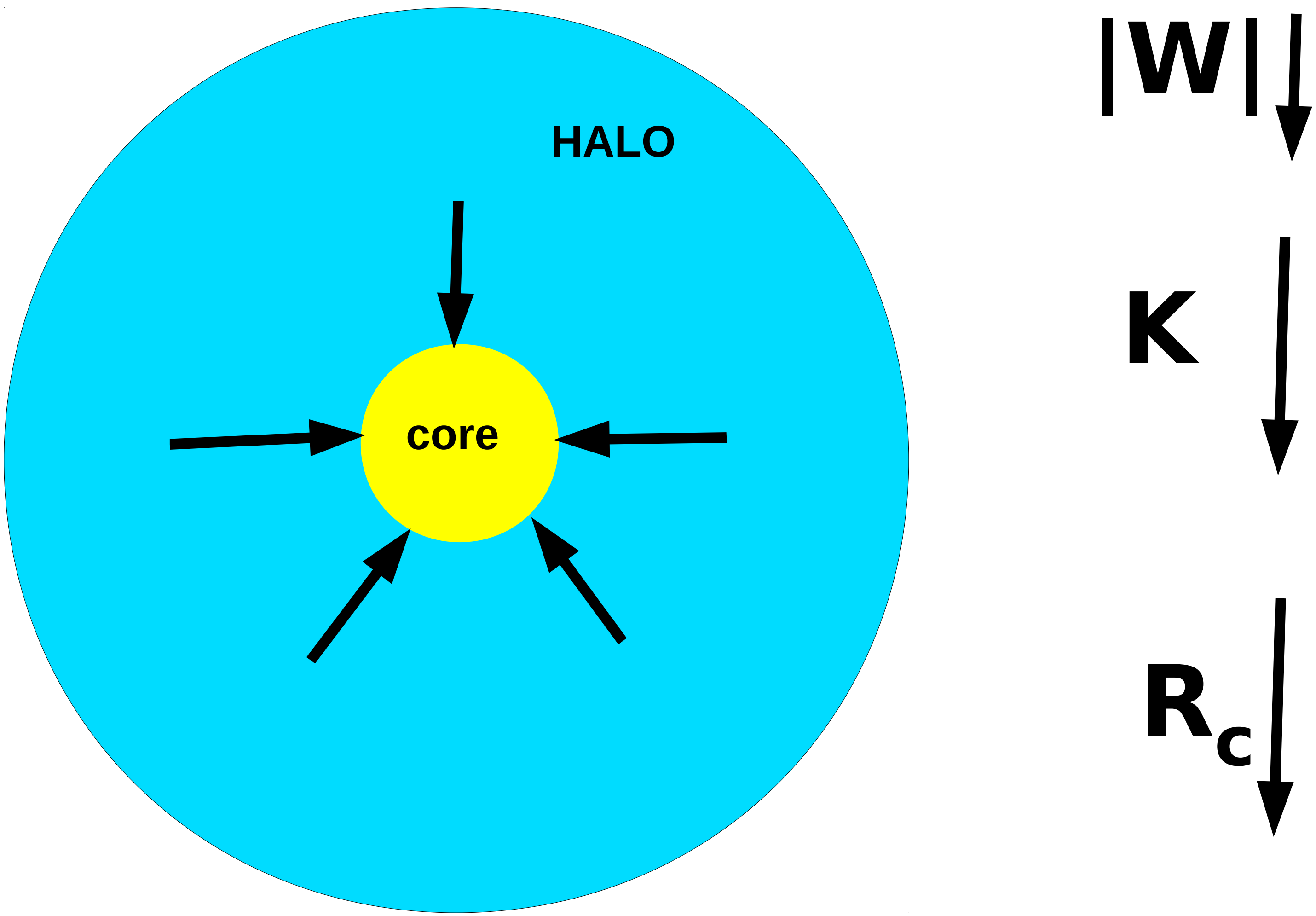
- two-body encounters are efficient →
leads to evaporation of the fastest stars from core



CORE COLLAPSE with CARTOONS

- leads to decrease of |W| and K – since 'fastest' stars are lost, the decrease in K is stronger than in |W|

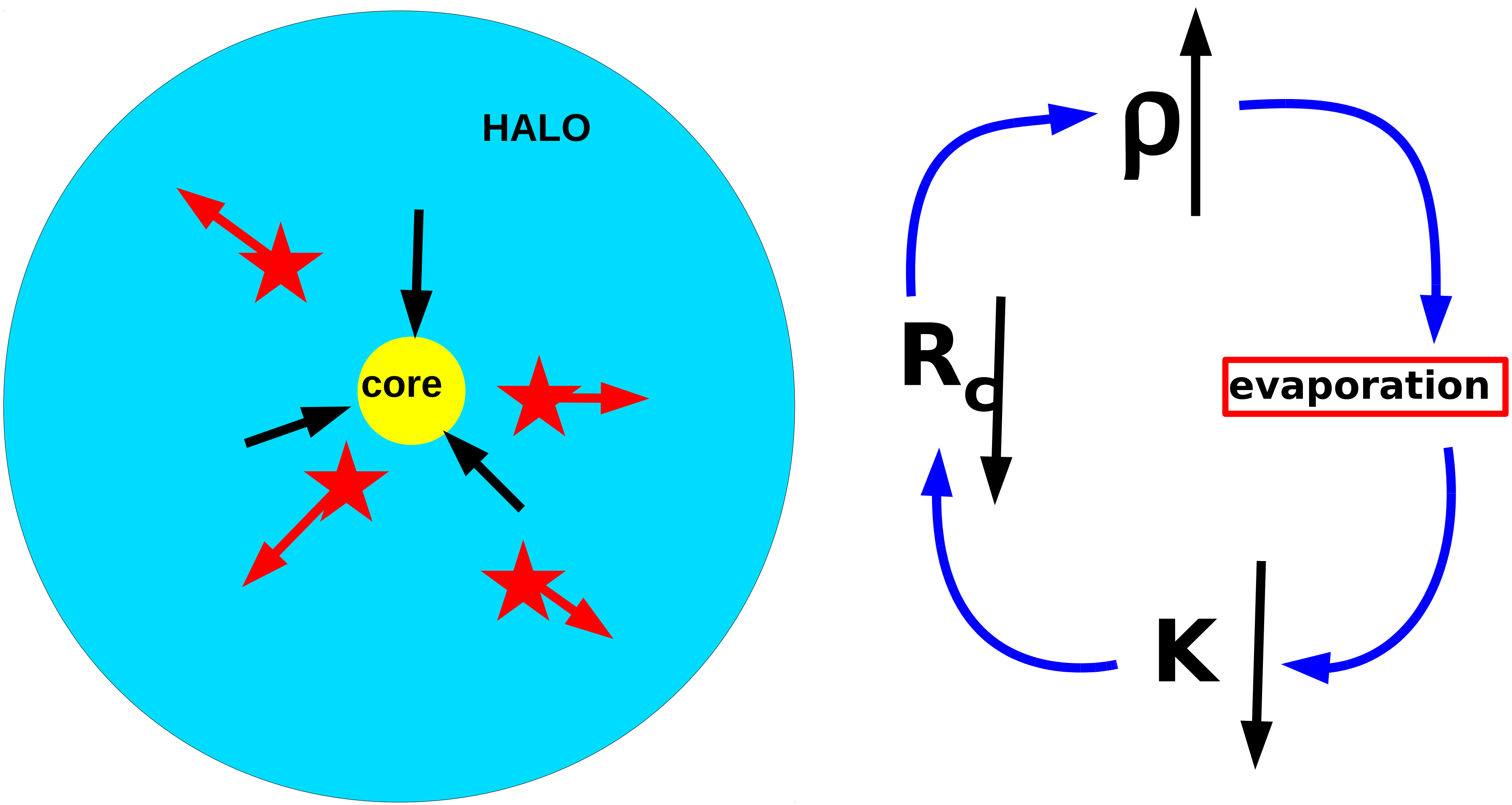
→ core contracts because |W| no longer balanced by K



CORE COLLAPSE with CARTOONS

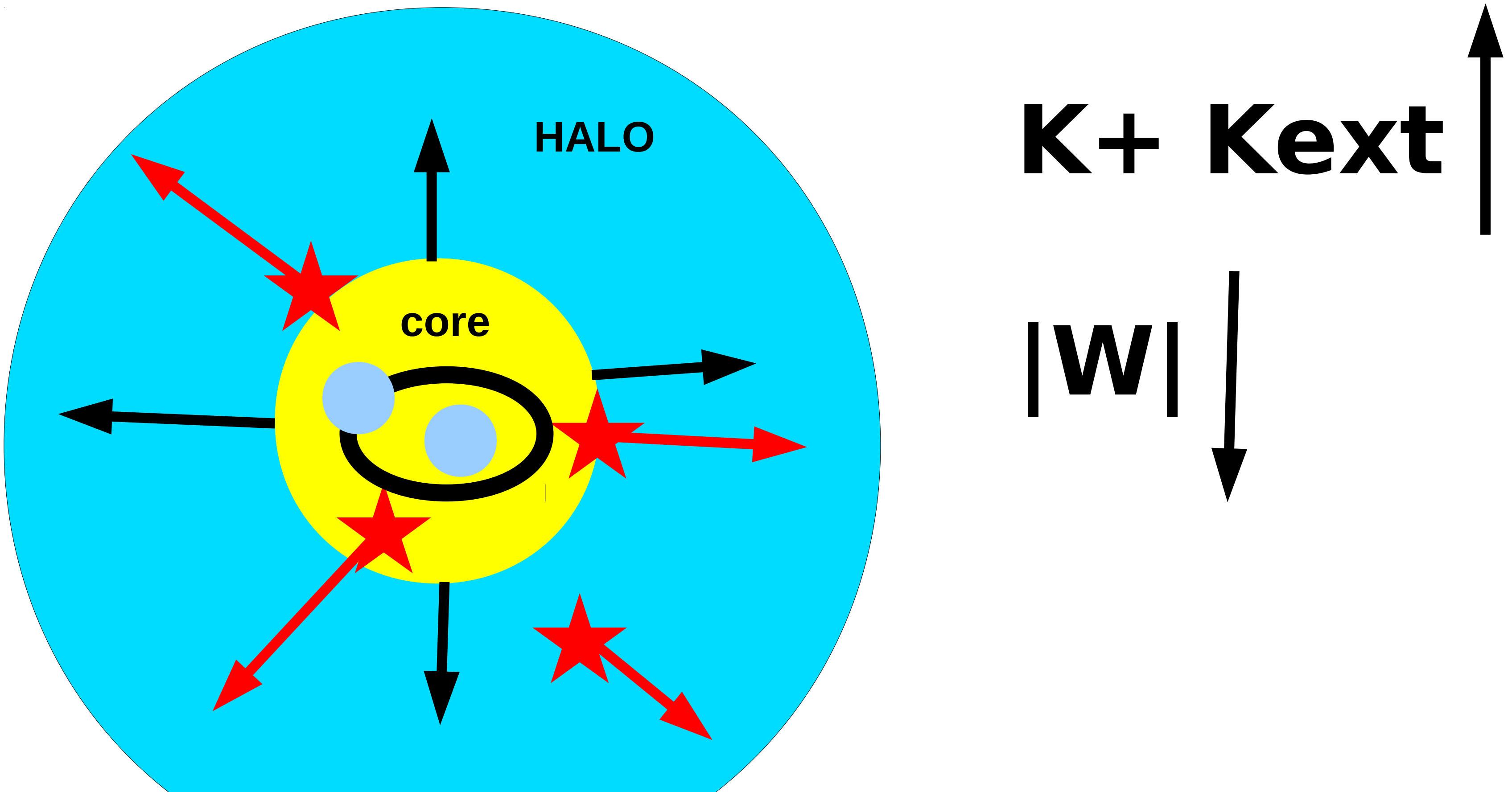
- **density increases and 2body increase → more fast stars evaporate, K decreases further, radius contracts more, ..**

- **RUNAWAY MECHANISM : core collapse!!!**



CORE COLLAPSE with CARTOONS

- NEEDS AN EXTERNAL SOURCE (Kext) TO BREAK IT:
    * 3body encounters: E extracted from binaries decreases |W| and increases K
    * Mass loss by stellar winds decrease |W|



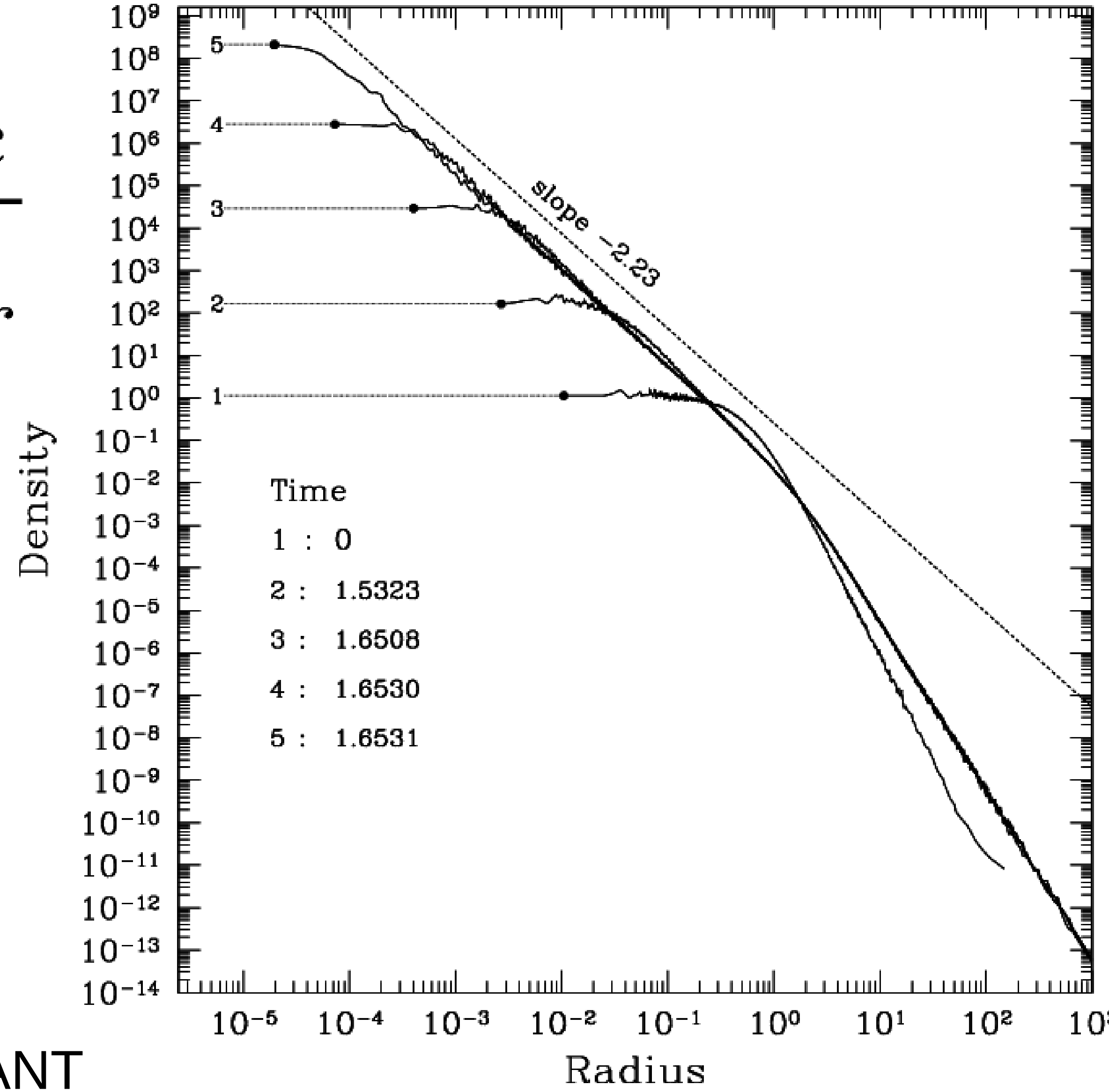
CORE COLLAPSE properties:

CORE COLLAPSE is SELF-SIMILAR (cfr. model of evaporation in slides 5-7: self-similarity is correct!)

$$\frac{d\rho_c}{dt} \rightarrow \mathrm{const}\,\frac{\rho_c}{t_{cr}}$$

central relaxation time

const ~ 3.6x10^-3 – 6x10^-3 from N-body simulations



Freitag & Benz 2001, A&A, 375, 711

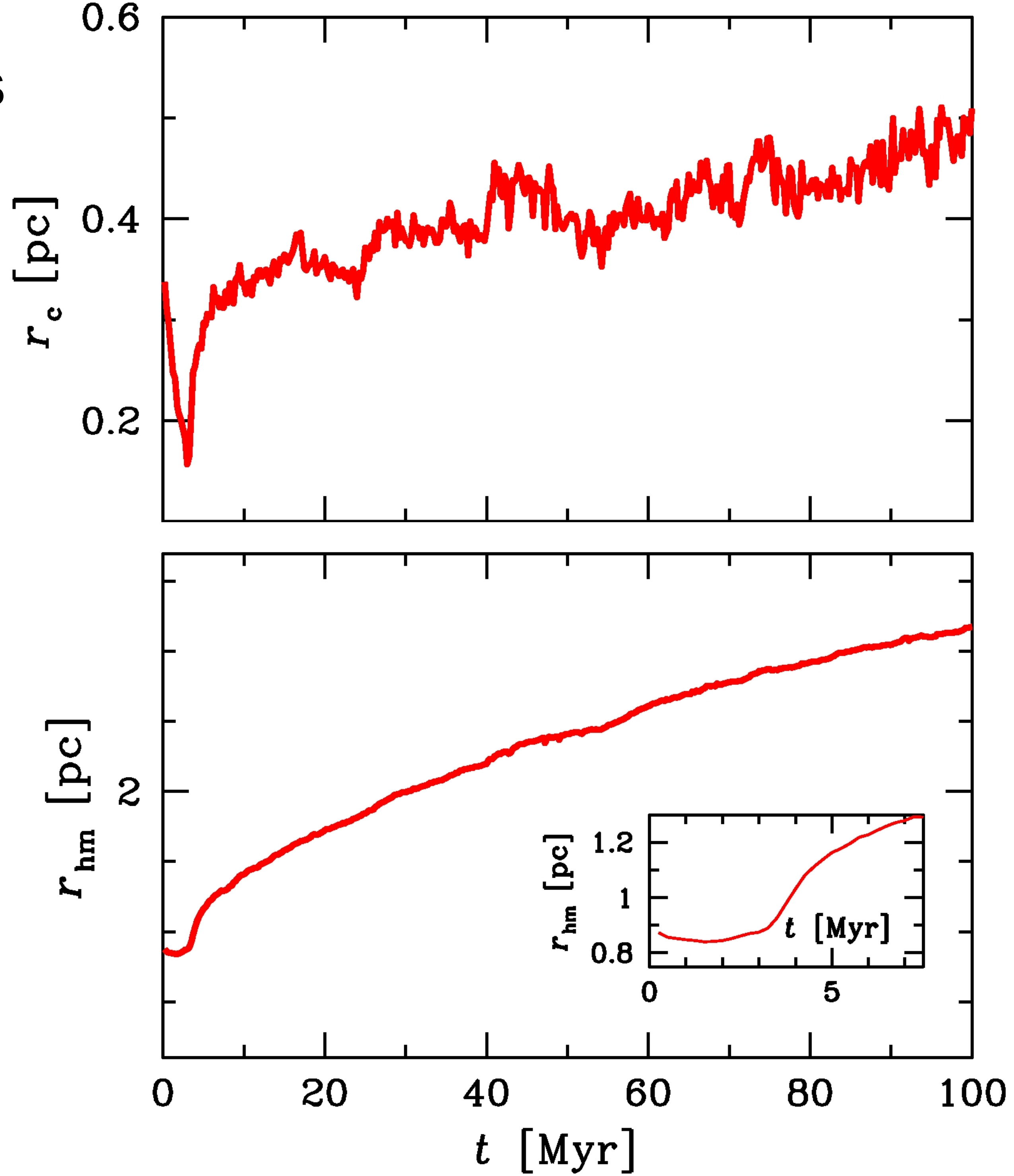
* DURING CORE COLLAPSE HALF-MASS RADIUS ~CONSTANT

POST CORE COLLAPSE PHASE:

CORE EXPANDS → **INJECTS ENERGY IN THE HALO IN FORM OF HIGH VELOCITY STARS** and matter

HALO is a good bath but not an ideal (i.e. perfect) bath: **HALO EXPANDS due to energy injection and also half-mass radius expands**

(Note: when speaking of half-mass radius, we refer mostly to the halo as core generally is << 1/10 of total mass)



POST CORE COLLAPSE PHASE:

HOW does halo expand?

(1) core collapse is self-similar half-mass relaxation time

$$t_{hm} \propto t$$

(2) from 1st lecture

$$t_{hm} \propto \frac{N}{\ln N} t_{cross} \sim N \, t_{cross}$$

(3) VIRIAL theorem

$$\frac{1}{2} M \langle v^2 \rangle = \frac{1}{2} \frac{G \, M^2}{r_{hm}}$$

(4)

$$t_{cross} = \frac{r_{hm}}{\langle v \rangle} \Rightarrow r_{hm}^3 \sim G \, M \, t_{cross}^2$$

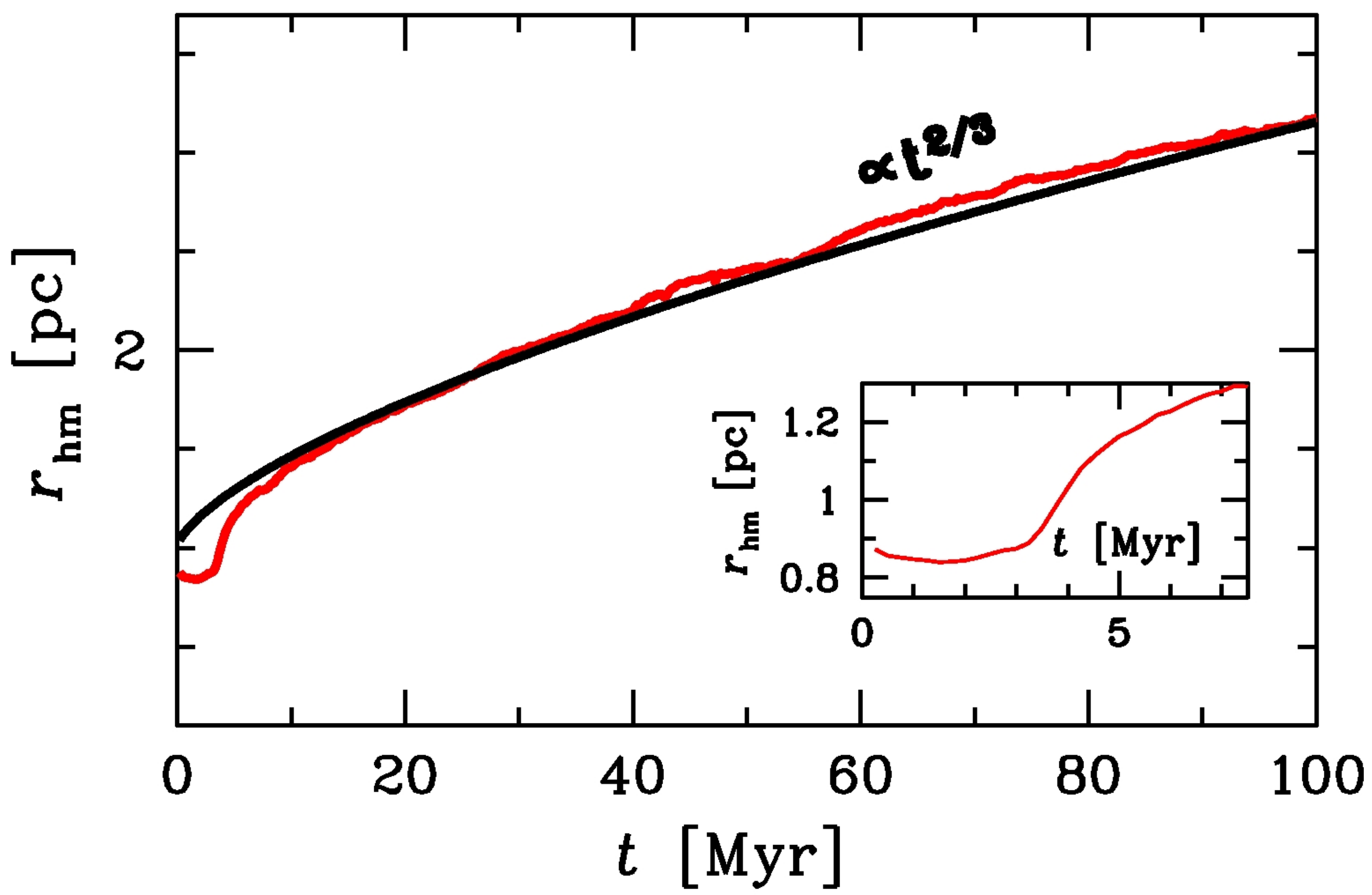
$$\Rightarrow \; t_{cross} \propto r_{hm}^{3/2} \quad \text{assuming } M \sim const$$

$$\Rightarrow r_{hm} \propto t_{cross}^{2/3} \propto t_{hm}^{2/3} \propto t^{2/3}$$

POST CORE COLLAPSE PHASE:

HOW does halo expand?

$$\Rightarrow r_{hm} \propto t_{cross}^{2/3} \propto t_{hm}^{2/3} \propto t^{2/3}$$



GRAVOTHERMAL OSCILLATIONS:

After first core collapse there may be a series of contractions/re-expansions of the core

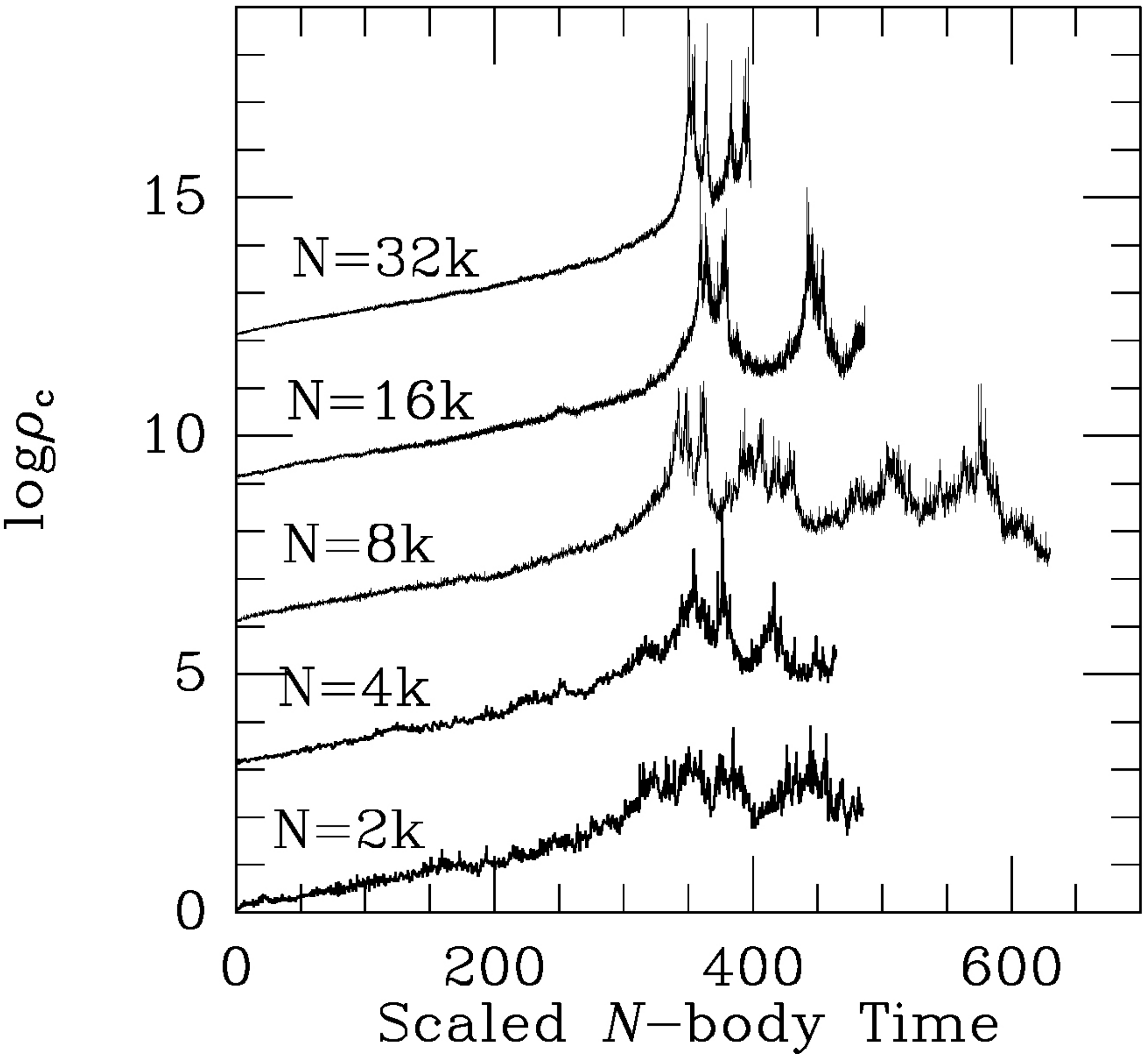
These are consequences of the fact that HEAT CAPACITY can be still negative

Note: only when *N*>10 000
why? Boh..

Hut 1997 (astro-ph/9704286) gives good idea:

For *N*<10 000, binaries are a steady engine.

For *N*>10000 the central density reached after 1st collapse is so high that no engine is sufficient to keep system stable after first bounce



GRAVOTHERMAL OSCILLATIONS:

There has been a lot of discussion whether

- secondary collapses are gravothermal
  i.e. are induced by negative heat capacity

  MOST LIKELY ANSWER: yes

- secondary reverses of collapse are gravothermal
  i.e. are induced by negative heat capacity
  reverse of the heating flow: when the bath becomes hotter than the core →
  heat keeps flowing from bath to core
  Core becomes cooler and cooler, and expands

  MOST LIKELY ANSWER: NO
  All reverses are due to 3-body encounters

WHY?

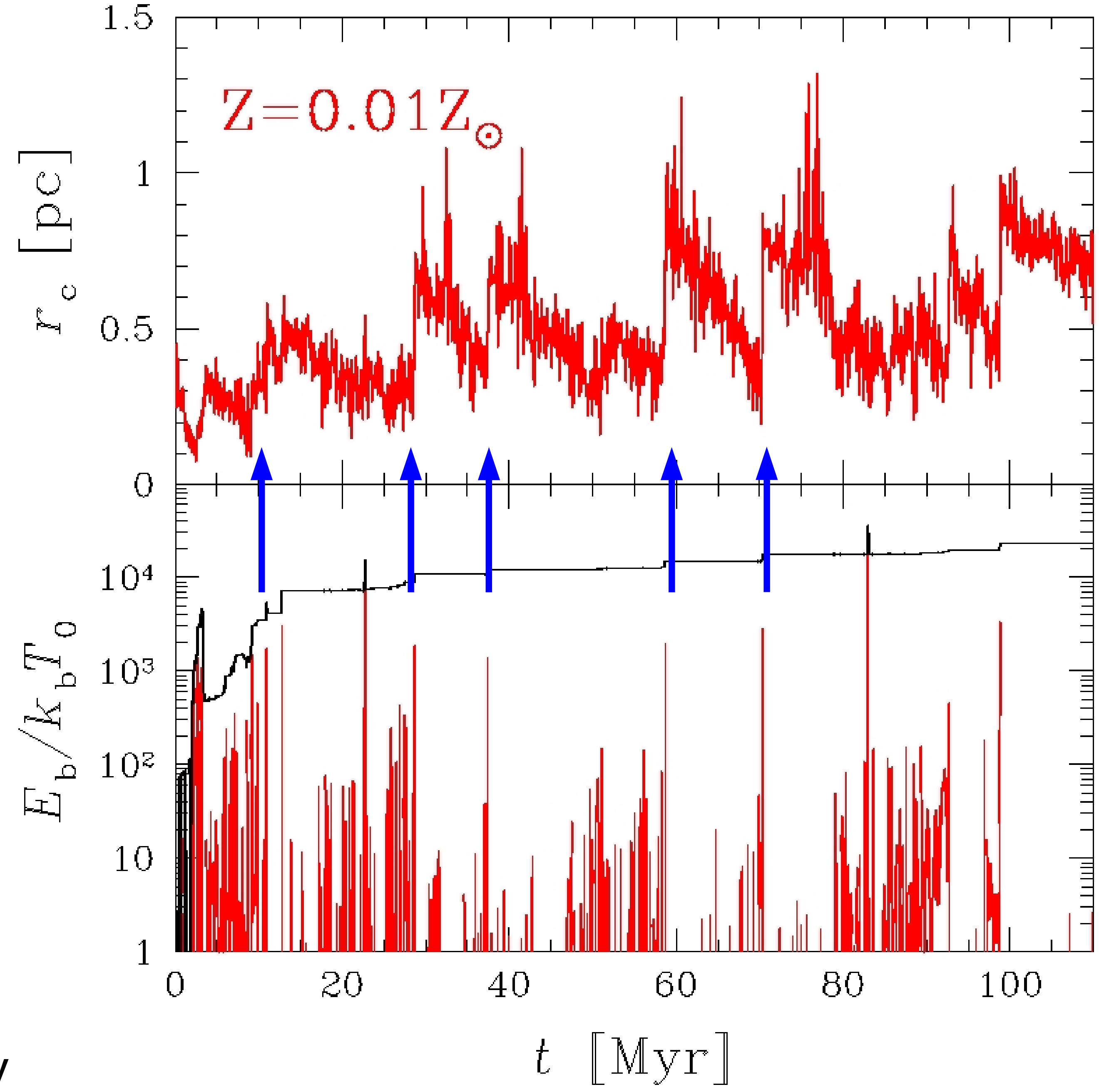
GRAVOTHERMAL OSCILLATIONS:

Core radius

Binary binding energy

Reverse of oscillations is not gravothermal because each reverse corresponds to a jump in binary binding energy

→ three-body encounters by binaries play the main role



Trani, MM & Bressan 2014

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

Processes described up to now (two-body relaxation, evaporation, gravothermal instability, core collapse and reversal) OCCUR EVEN IF STARS ARE EQUAL MASS

BUT stars form with a mass spectrum

The most important effects of unequal-mass system are MASS SEGREGATION and SPITZER'S INSTABILITY

EQUIPARTITION: even collisional systems (i.e. where two-body relaxation is efficient) subject to gravity evolve to satisfy equipartition theorem of statistical mechanics, i.e.
PARTICLES TEND TO HAVE THE SAME AVERAGE KINETIC ENERGY

Thus, equipartition occurs EVEN if stars are equal mass.

If stars are equal mass → equipartition implies that have the same average VELOCITY

$$k_i = \frac{1}{2} m \, v_i^2$$

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

If stars are equal mass → equipartition implies that have the same average VELOCITY

$$k_i = \frac{1}{2} m\, v_i^2$$

If particles have different masses, this has a relevant consequence:

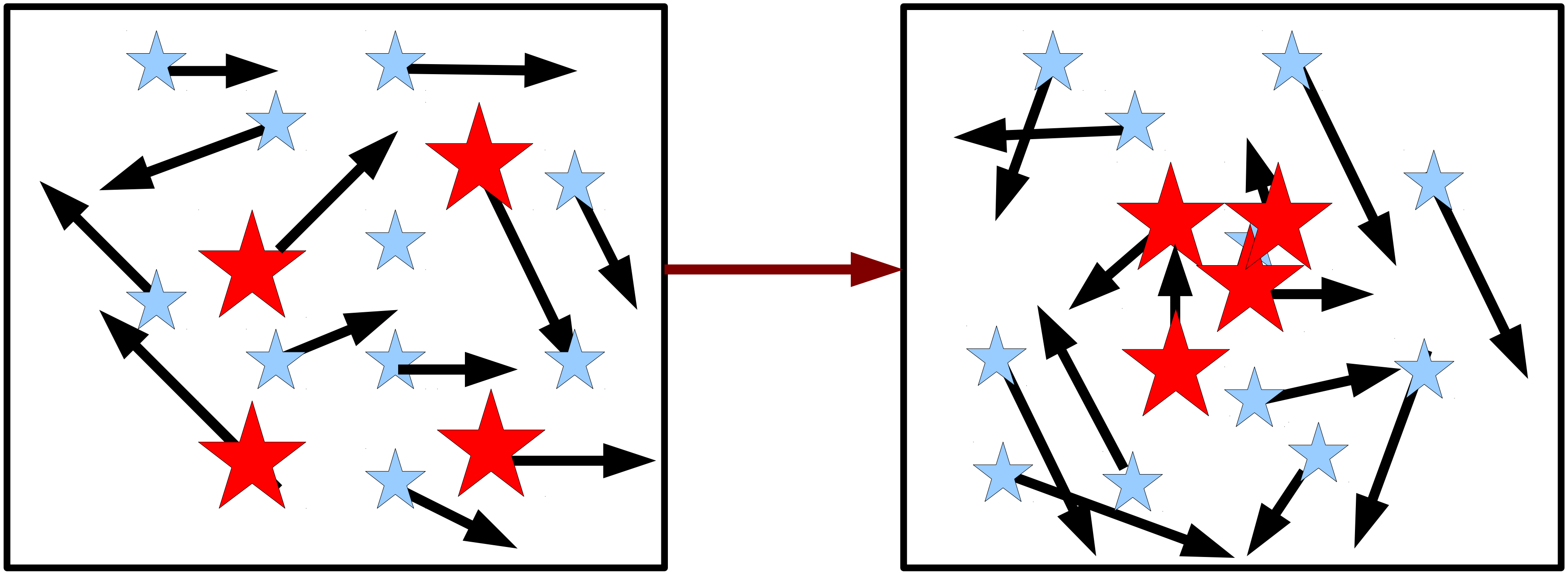
$$m_i \langle v_i^2 \rangle = m_j \langle v_j^2 \rangle$$

$$\text{if} \quad m_i > m_j \Rightarrow \langle v_i^2 \rangle < \langle v_j^2 \rangle$$

During two-body encounters, massive stars transfer kinetic energy to light stars. Massive stars slow down, light stars move to higher velocities.

Equipartition in multi-mass systems is reached via ***dynamical friction***

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY



MASS SEGREGATION

During two-body encounters, massive stars transfer kinetic energy to light stars. Massive stars slow down, light stars move to higher velocities.

This means that **heavier stars drift to the centre of the cluster**, producing **MASS SEGREGATION**
(i.e. local mass function different from IMF)

Equipartition in multi-mass systems is reached via ***dynamical friction***

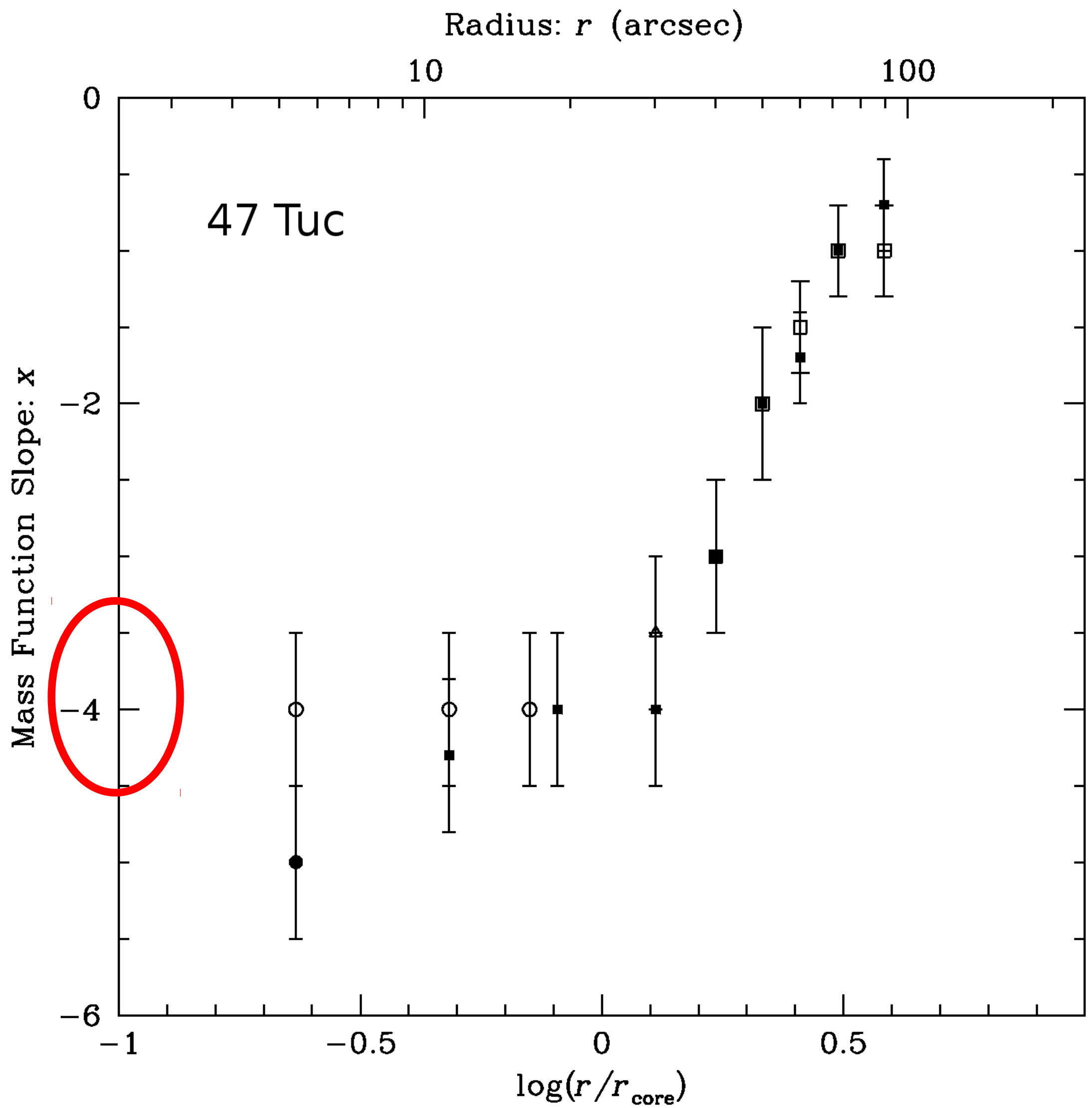
(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

This means that **heavier stars drift to the centre of the cluster**, producing **MASS SEGREGATION** (i.e. local mass function different from IMF)

e.g. core of 47 Tucanae
(Monkman et al. 2006, ApJ, 650, 195)
x:= mass function slope

$$dN/dm = m^{-(1+x)}$$

where Salpeter x=+1.35

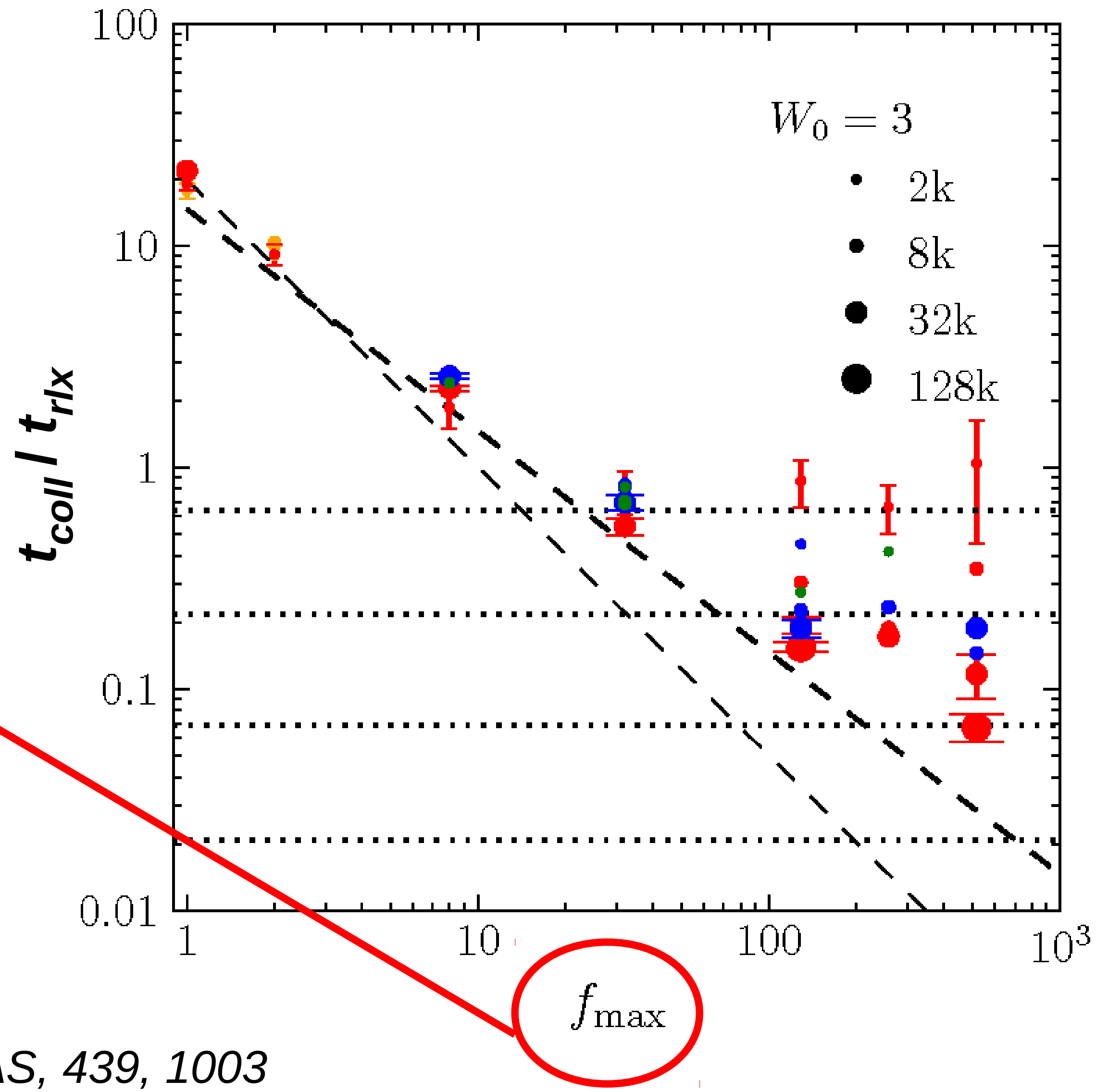


(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

This means that **heavier stars drift to the centre of the cluster**, producing **MASS SEGREGATION** (i.e. local mass function different from IMF)

→ MASS SEGREGATION increases the instability of the system and induces a **FASTER COLLAPSE (t_{coll}~0.2 t_{rlx} rather than t_{coll}~15 t_{rlx})**.

$$f_{\rm max} \equiv \frac{m_{\rm max}}{\langle m \rangle}$$



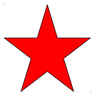
Fujii & Portegies Zwart et al. 2014, MNRAS, 439, 1003

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

SPITZER INSTABILITY **(or mass stratification instability):**

It is not always possible to reach equipartition in a multi-mass system.

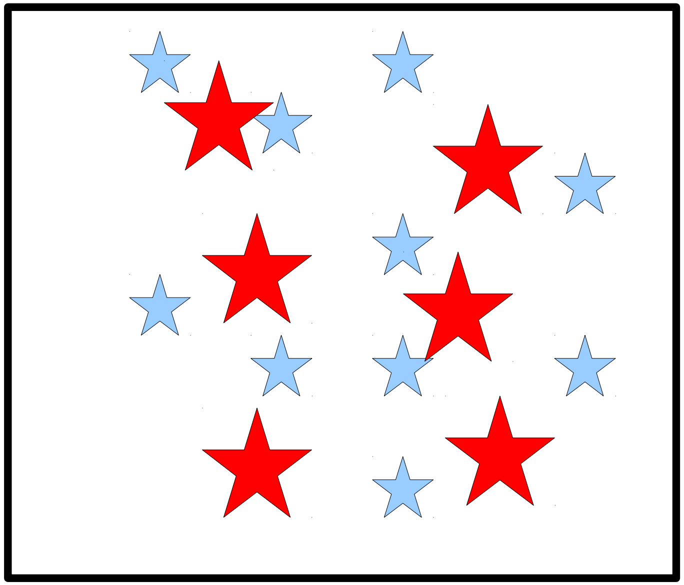
Let us suppose that there are two populations with two different masses:



HEAVY POPULATION
m_2 (total mass M_2)

LIGHT POPULATION
m_1 (total mass M_1)

$m_2 > m_1$



$M_2 \sim M_1$

If the total mass of the heavy population is similar to the total mass of the light population, equipartition is not possible:

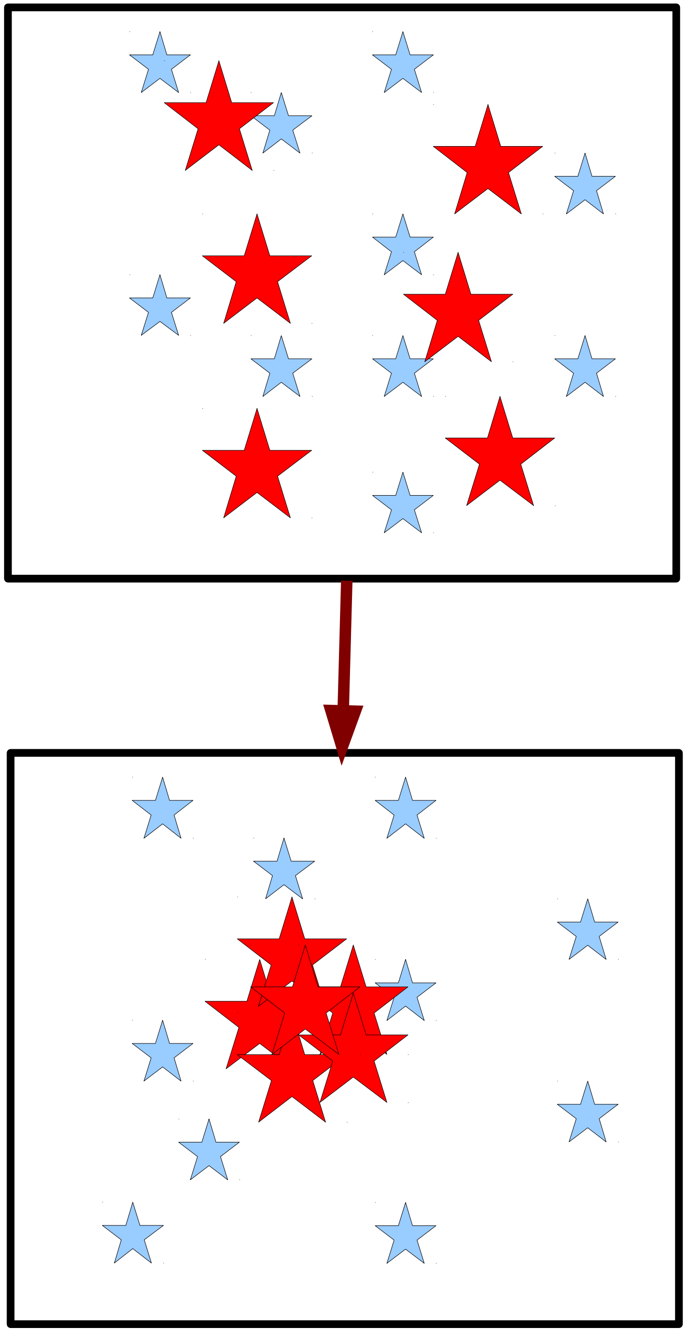
$M_2 \langle v_2^2 \rangle >> M_1 \langle v_1^2 \rangle$

THE LIGHT POPULATION CANNOT ABSORB ALL THE KINETIC ENERGY THAT MUST BE TRANSFERRED FROM THE HEAVY POPULATION TO REACH EQUIPARTITION

(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

SPITZER INSTABILITY:

The heavy population forms a
CLUSTER WITHIN THE CLUSTER
(sub-cluster at the centre of the cluster),
DYNAMICALLY DECOUPLED
from the rest of the cluster.



The massive stars in the sub-cluster keep transferring kinetic energy to the lighter stars but cannot reach equipartition:
the core of massive stars continues to
CONTRACT TILL INFINITE DENSITY!

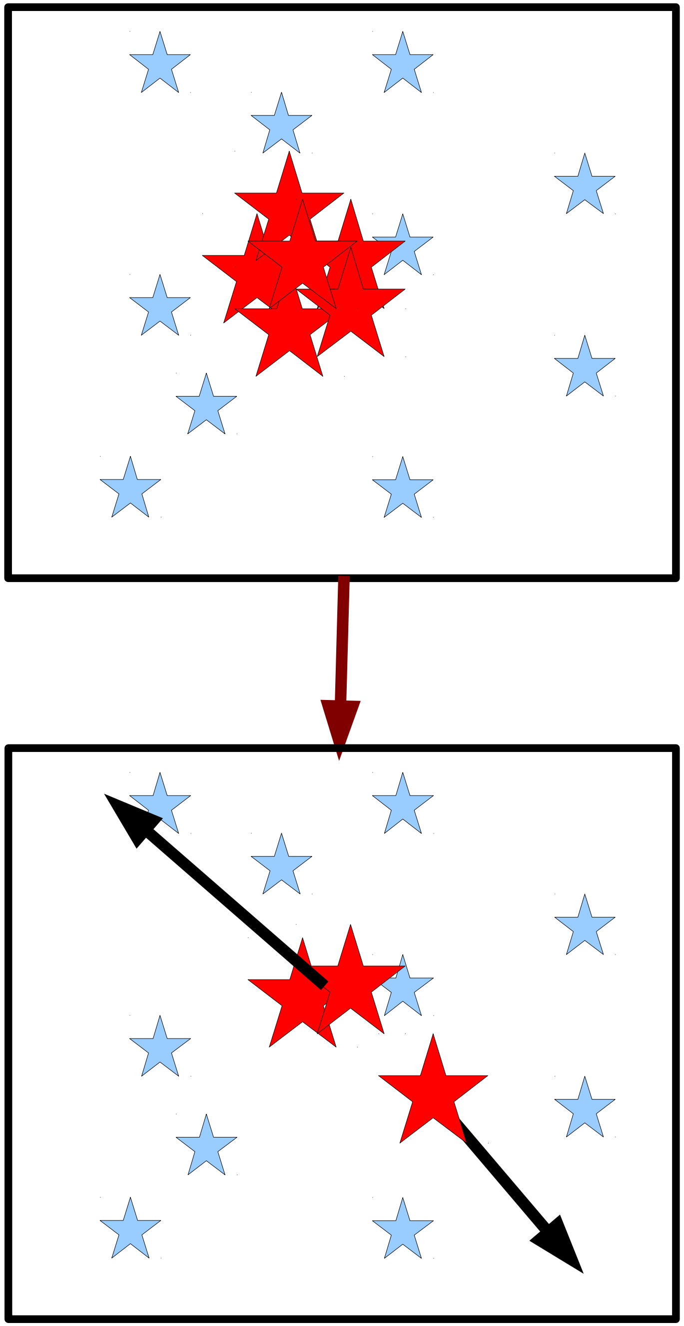
(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

SPITZER INSTABILITY:

The contraction stops

– when most of the massive stars eject each-other from the SC by 3-body encounters

SPITZER INSTABILITY ENHANCES THE EJECTION OF MASSIVE OBJECTS (E.G. BLACK HOLES) FROM SCs !!!!



(3) EQUIPARTITION, MASS SEGREGATION AND SPITZER'S INSTABILITY

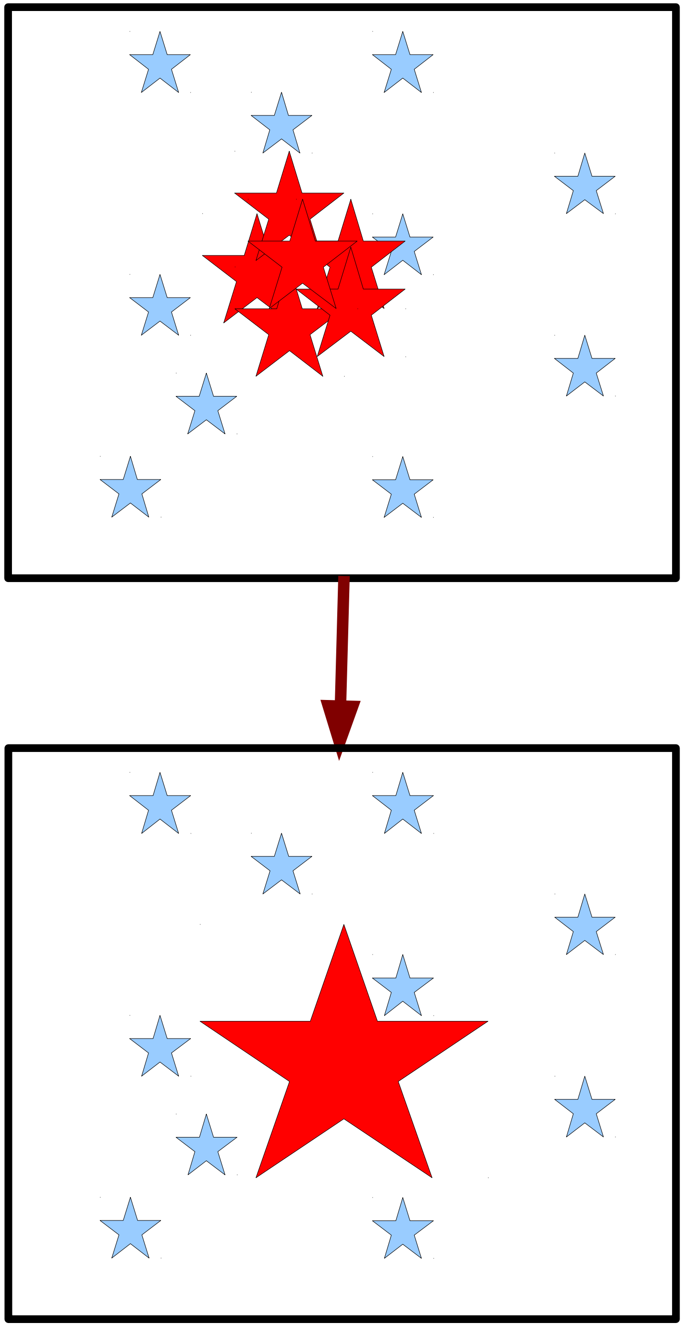
SPITZER INSTABILITY:

The contraction stops

– when most of the massive stars eject each-other from the SC by 3-body encounters

SPITZER INSTABILITY ENHANCES THE EJECTION OF MASSIVE OBJECTS (E.G. BLACK HOLES) FROM SCs !!!!

– or when most of the massive stars collapse into a single object



(3) SPITZER'S INSTABILITY

SPITZER'S INSTABILITY (or mass stratification instability):
It is not always possible to reach equipartition in a multi-mass.

Let us suppose that there are two populations with two different masses: m_1 (total mass M_1) and m_2 (total mass M_2), with $m_1 < m_2$.

We explore 2 limit cases where equipartition is impossible.

1) $M_2 >> M_1 \Rightarrow$ potential is dominated by massive stars

$\Rightarrow$ *$\langle v^2 \rangle$ of the massive stars is ~ ¼ $\langle v_{esc}^2 \rangle$*

$\Rightarrow$ *if $m_2/m_1 > 4$, the $\langle v^2 \rangle$ of light stars is higher than $\langle v_{esc}^2 \rangle$*

$\Rightarrow$ ***ALL LIGHT STARS EVAPORATE FROM THE CLUSTER!!!***

Not very important in practice because IMF is not sufficiently top-heavy

(3) SPITZER'S INSTABILITY

SPITZER'S INSTABILITY:

2) $M_2 \sim M_1$ (the case of the so called Spitzer's instability)

If the total mass of the heavy population is similar to the total mass of the light population, equipartition is not possible:
the heavy population forms a cluster within the cluster,
i.e. a sub-cluster at the centre of the cluster,
dynamically decoupled from the rest of the cluster.
The sub-cluster of the heavy population tends to contract.

DEMONSTRATION:
(Note that I did not put numerical coefficients & simplified!)
(a) Assume that there are two populations (1 and 2) with $m_2 >> m_1$
(b) assume total mass $M_2 < M_1$
(c) assume $M_1(r) \sim \rho_{01}\, r^3$ (ρ_{01}:= initial density of population 1)

$\rho_{m2} \sim M_2/r_2^3$ *(ρ_{m2}:= average density of population 2, r_2:=half mass radius of population 2)*

$\rho_{m1} \sim M_1/r_1^3$ *(ρ_{m1}:= average density of population 1, r_1:=half mass radius of population 1)*

(3) SPITZER'S INSTABILITY

From equipartition: $$m_i \left\langle v_i^2 \right\rangle = m_j \left\langle v_j^2 \right\rangle \qquad (1)$$

From virial theorem: $$\left\langle v_2^2 \right\rangle \sim \frac{G\,M_2}{r_2} + \frac{G}{M_2} \int_0^\infty \frac{\rho_2\,M_1(\vec{r})}{\vec{r}} dV \qquad (2)$$

$$\left\langle v_1^2 \right\rangle \sim \frac{G\,M_1}{r_1} + \frac{G}{M_1} \int_0^\infty \frac{\rho_1\,M_2(\vec{r})}{\vec{r}} dV \qquad (3)$$

$M_1 \gg M_2(\vec{r})$ → 0

Substituting (2) and (3) into (1) and using the assumptions a, b and c:

$$m_2 \left(\frac{G\,M_2}{r_2} + \frac{G\,\rho_{m2}}{M_2} \int \rho_{01} \frac{\vec{r}^3}{\vec{r}} dV \right) = m_1 \frac{G\,M_1}{r_1}$$

$$m_2 \left[\frac{M_2}{\left(\frac{M_2}{\rho_{m2}}\right)^{1/3}} + \frac{\rho_{m2}}{M_2} \int \rho_{01}\,\vec{r}^2\,dV \right] = m_1 \frac{M_1}{\left(\frac{M_1}{\rho_{m1}}\right)^{1/3}}$$

(3) SPITZER'S INSTABILITY

$$m_2\, M_2^{2/3}\, \rho_{m2}^{1/3} \left(1 + \frac{\rho_{m2} \int \rho_{01}\, \vec{r}^2\, dV}{M_2^{5/3}\, \rho_{m2}^{1/3}}\right) = m_1\, M_1^{2/3}\, \rho_{m1}^{1/3}$$

$$\frac{m_2}{m_1} \left(\frac{M_2}{M_1}\right)^{2/3} = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/3}}{\left[1 + \frac{\rho_{m2} \int \rho_{01}\, \vec{r}^2\, dV}{\left(\rho_{m2}\, r_2^3\right)^{5/3}\, \rho_{m2}^{1/3}}\right]}$$

$$\frac{m_2}{m_1} \left(\frac{M_2}{M_1}\right)^{2/3} = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/3}}{\left(1 + \frac{\rho_{m2} \int \rho_{01}\, \vec{r}^2\, dV}{\rho_{m2}^2\, r_2^5}\right)}$$

$$\frac{m_2}{m_1} \left(\frac{M_2}{M_1}\right)^{2/3} = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/3}}{\left[1 + \frac{\rho_{01}}{\rho_{m2}} \left(\frac{r_2}{r_2}\right)^5\right]}$$

(3) SPITZER'S INSTABILITY

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{(\rho_{m1}/\rho_{m2})^{1/2}}{\left[1 + \frac{\rho_{01}}{\rho_{m2}}\right]^{3/2}}$$

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{(\rho_{m1}/\rho_{m2})^{1/2}}{\left[1 + \frac{\rho_{m1}}{\rho_{m1}}\frac{\rho_{01}}{\rho_{m2}}\right]^{3/2}}$$

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{(\rho_{m1}/\rho_{m2})^{1/2}}{\left[1 + \frac{\rho_{01}}{\rho_{m1}} \left(\frac{\rho_{m1}}{\rho_{m2}}\right)\right]^{3/2}}$$

(3) SPITZER'S INSTABILITY

$$\left(\frac{m_2}{m_1}\right)^{3/2} \left(\frac{M_2}{M_1}\right) = \frac{\left(\rho_{m1}/\rho_{m2}\right)^{1/2}}{\left(1+\alpha\,\frac{\rho_{m1}}{\rho_{m2}}\right)^{3/2}}$$

Our simplified α :

$$\alpha \equiv \frac{\rho_{01}}{\rho_{m1}}$$

Spitzer's α :

$$\alpha \equiv \frac{5\,\rho_{01}}{4\,\rho_{m1}} \left(\frac{r_{2s}}{r_2}\right)^2 \sim 5.6$$

where r_{2s}^2 is the mean value of r^2 for the population 2

Maximum possible value for the right-hand term ~ 0.16

$$\Rightarrow \frac{M_2}{M_1} < 0.16 \left(\frac{m_2}{m_1}\right)^{3/2}$$

OTHERWISE EQUIPARTITION CANNOT BE REACHED!

(3) SPITZER'S INSTABILITY

SPITZER'S INSTABILITY:
It is not possible to reach equipartition if $M_2/M_1 < 0.16\,(m_2/m_1)^{3/2}$.

If the total mass of the heavy population is similar to the total mass of the light population, equipartition is not possible:

the heavy population forms a cluster within the cluster,
i.e. a sub-cluster at the centre of the cluster,
dynamically decoupled from the rest of the cluster.

The massive stars in the sub-cluster keep transferring kinetic energy to the lighter stars but cannot reach equipartition: the core of massive stars continues to contract till infinite density!

The contraction stops when most of the massive stars eject each-other from the cluster by 3-body encounters (see next lecture) or when most of the massive stars collapse into a single object (see last lecture).

TIMESCALES FOR RELAXATION and CORE COLLAPSE in different SCs

Table 1.3. Time scales

Time scale	symbol	bulge	globular	YoDeC	Open cluster
Star	t_{ms}	10Gyr	10Gyr	10Myr	10Myr
size	R	100pc	10pc	$\lesssim$ 1pc	10pc
mass	M	$10^9 M_\odot$	$10^6 M_\odot$	$10^5 M_\odot$	$1000 M_\odot$
velocity	$\langle v \rangle$	100km s^{-1}	10km s^{-1}	10km s^{-1}	1km s^{-1}
relaxation	t_{rt}	10^{15}yr	3 Gyr	50Myr	100Myr
crossing	t_{hm}	100Myr	10 Myr	100Kyr	1Myr
t_{rt}/t_{ms}		10^5	3	5	10
t_{hm}/t_{ms}		0.01	1	10^{-4}	0.1

From Portegies Zwart 2004, astro-ph/0406550

Note: t_{coll}~0.2 t_{rlx}

Young dense star clusters (YoDeC) are the only clusters with relaxation and core collapse time of the same order of magnitude as massive star evolution

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